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SOUSTAVY OBYČEJNÝCH DIFERENCIÁLNÍCH ROVNIC PRVNÍHO ŘÁDU (SODR1Ř)

tvar:

$$\frac{dy_1}{dt} = y_1' = a_{11}(x)y_1 + a_{12}(x)y_2 + \dots + a_{1n}(x)y_n + f_1(x)$$

$$\frac{dy_2}{dt} = y_2' = a_{21}(x)y_1 + a_{22}(x)y_2 + \dots + a_{2n}(x)y_n + f_2(x)$$

$$\dots$$

$$\frac{dy_n}{dt} = y_n' = a_{n1}(x)y_1 + a_{n2}(x)y_2 + \dots + a_{nn}(x)y_n + f_n(x)$$

→ fce $a_{ij}(x)$ a $f_k(x)$ jsou spojité na intervalu I

→ je-li $f_k(x) = 0$ $\forall k = 1, 2, \dots, n$ Homogenní soustava LODR1

$f_k(x) \neq 0$ pro nějaké $k = 1, 2, \dots, n$ Nehomogenní soustava LODR1

→ Exaktní řešení umíme pouze pro $a_{ij}(x)$ konstantní, jinak pouze numericky

Maticový zápis:

$$A = \begin{pmatrix} a_{11}(x) & a_{12}(x) & \dots & a_{1n}(x) \\ a_{21}(x) & a_{22}(x) & \dots & a_{2n}(x) \\ \dots & & & \\ a_{n1}(x) & a_{n2}(x) & \dots & a_{nn}(x) \end{pmatrix} \quad y' = \begin{pmatrix} y_1' \\ y_2' \\ \vdots \\ y_n' \end{pmatrix} \quad y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \quad f(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_n(x) \end{pmatrix}$$

$$\Rightarrow y' = A \cdot y + f(x)$$

→ speciálně

$$A = \begin{pmatrix} a_{11}(x) & a_{12}(x) \\ a_{21}(x) & a_{22}(x) \end{pmatrix} \quad y' = \begin{pmatrix} y_1' \\ y_2' \end{pmatrix} \quad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \quad f(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix}$$

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} a_{11}(x) & a_{12}(x) \\ a_{21}(x) & a_{22}(x) \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix} \rightarrow y' = Ay + f$$

→ numericky → vyšší řády DR převádíme na systém SODR1Ř

Př: a) $y'' + 2y' + y = 0$

substituce: $(y = y_1) \rightarrow y' = y_1' = y_2$

$(y' = y_2) \rightarrow y'' = y_2'$

DDZ: $y_2' + 2y_2 + y_1 = 0$

$\rightarrow y_2' = -y_1 - 2y_2$

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= -y_1 - 2y_2 \end{aligned} \rightarrow \text{maticově} \begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

b) $y'' + 2y' + y = f(x)$

DDZ: $y_2' + 2y_2 + y_1 = f(x)$

$y_2' = -y_1 - 2y_2 + f(x)$

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= -y_1 - 2y_2 + f(x) \end{aligned} \rightarrow \text{maticově} \begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ f(x) \end{pmatrix}$$

→ párový x nepárový systém ;

párový : $x'(t) = x(t) - y(t) + 2t - t^2 - t^3$; $x(0)=1$
$y'(t) = x(t) + y(t) - 4t^2 + t^3$; $y(0)=0$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 2t - t^2 - t^3 \\ -4t^2 + t^3 \end{pmatrix} ; \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

nepárový : $x'(t) = x(t) + 2t - t^2 - t^3$. $x(0)=1$
$y'(t) = y(t) - 4t^2 + t^3$ $y(0)=0$

$$\underbrace{\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 2t - t^2 - t^3 \\ -4t^2 + t^3 \end{pmatrix} ; \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

diagonální matice → můžeme řešit jednotlivě jako dva nezávislé počáteční problémy

→ Použít lze stejné metody : Eulerova, Modifikovaná Eulerova metoda; RK 2. řádu, RK 4. řádu ...

→ jen maticově ←

Soustavy SODR 1Ř   EULEROVA METODA

Př 1 $y'' + y = 0 \;;\; y(0) = 2 \;;\; y'(0) = 3$

exaktní řešení
($OŘ$: $y(x) = C_1 \cos x + C_2 \sin x$
$PŘ$: $y(x) = 2\cos x + 3\sin x$)

$x \in \langle 0; \frac{\pi}{2} \rangle \qquad n = 10 \rightarrow h = \frac{\frac{\pi}{2} - 0}{10} = \frac{\pi}{20}$

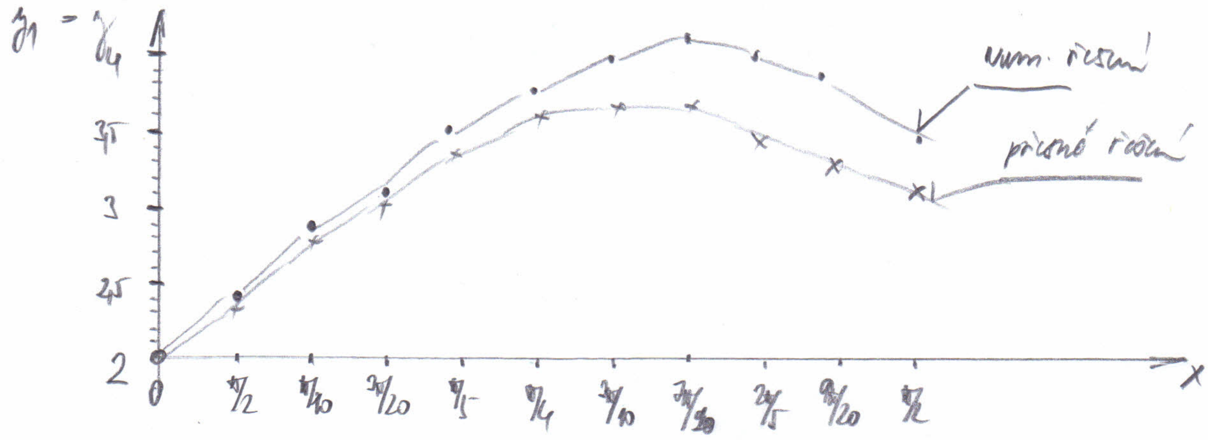
$(y = y_1)' \Rightarrow y' = y_1' = y_2$
$(y' = y_2)' \rightarrow y'' = y_2'$

DDZ: $y_2' + y_1 = 0$
$y_2' = -y_1$

$$\left.\begin{aligned} y_1' &= y_2 \\ y_2' &= -y_1 \end{aligned}\right\} \quad \begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

EV: $y_1(i+1) = y_1(i) + \frac{\pi}{2} y_2(i) \quad (\Rightarrow y_1(i+1) = A + \frac{\pi}{20} B)$
$y_2(i+1) = y_2(i) - \frac{\pi}{2} y_1(i) \quad (\Rightarrow y_2(i+1) = B - \frac{\pi}{20} A)$

i	x_i		$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$		$y = y_1$ EXAKTNÍ
0	0	C	$\begin{pmatrix} 2 \\ 3 \end{pmatrix}$	A B	2
1	$\pi/20$	C	$\begin{pmatrix} 2{,}4712 \\ 2{,}6852 \end{pmatrix}$	D→A B	2,4447
2	$\pi/10$	C	$\begin{pmatrix} 2{,}8931 \\ 2{,}2977 \end{pmatrix}$	D→A B	2,8292
3	$3\pi/20$	C	$\begin{pmatrix} 3{,}254 \\ 1{,}8432 \end{pmatrix}$	D→A B	3,144
4	$\pi/5$	C	$\begin{pmatrix} 3{,}5436 \\ 1{,}3321 \end{pmatrix}$	D→A B	3,3814
5	$\pi/4$	C	$\begin{pmatrix} 3{,}7528 \\ 0{,}7754 \end{pmatrix}$	D→A B	3,5355
6	$3\pi/10$	C	$\begin{pmatrix} 3{,}8746 \\ 0{,}1859 \end{pmatrix}$	D→A B	3,6026
7	$7\pi/20$	C	$\begin{pmatrix} 3{,}9038 \\ -0{,}4227 \end{pmatrix}$	D→A B	3,581
8	$2\pi/5$	C	$\begin{pmatrix} 3{,}8374 \\ -1{,}0359 \end{pmatrix}$	D→A B	3,4712
9	$9\pi/20$	C	$\begin{pmatrix} 3{,}6747 \\ -1{,}6387 \end{pmatrix}$	D→A B	3,2759
10	$\pi/2$		$\begin{pmatrix} 3{,}4173 \\ -2{,}2159 \end{pmatrix}$		3



DU2 $y'' - y' - 2y = 4x^2$; $y(0) = 1$; $y'(0) = 4$; $x \in \langle 0; 2 \rangle$; $n = 8$; $h = \frac{2-0}{8} = 0{,}25$

(Př: $y = 2e^{-x} + 2e^{2x} - 2x^2 + 2x - 3$)

$(y = y_1)' \Rightarrow y' = y_1' = y_2$ DDR: $y_2' - y_2 - 2y_1 = 4x^2$

$(y' = y_2)' \Rightarrow y'' = y_2'$ $\quad y_2' = 2y_1 + y_2 + 4x^2$

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 4x^2 \end{pmatrix}$$

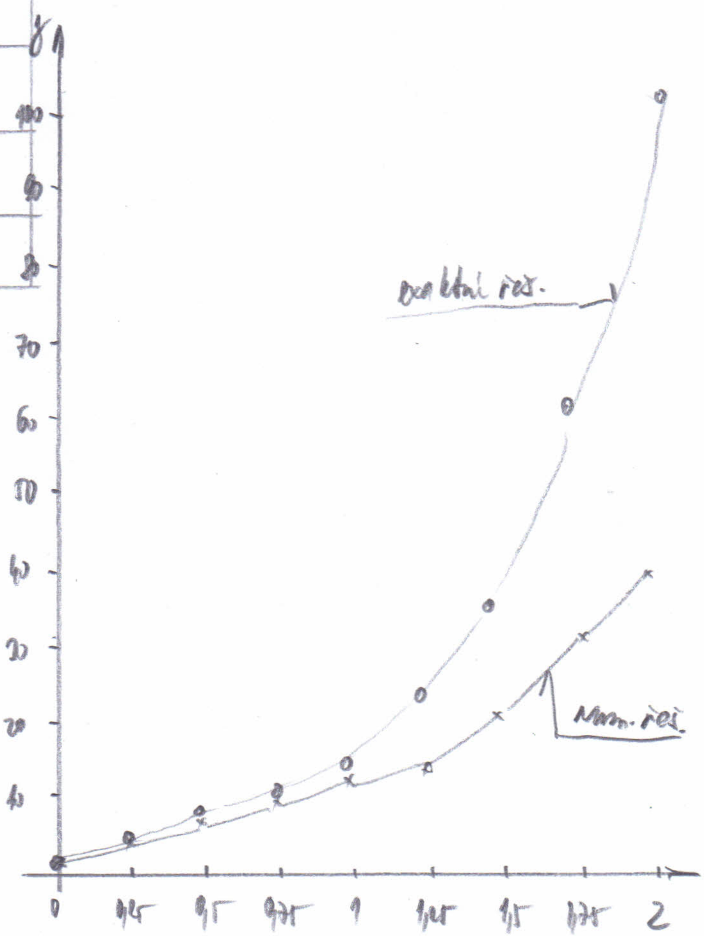
IV: $y_1(i+1) = y_1(i) + 0{,}25 \cdot y_2(i)$

$y_2(i+1) = y_2(i) + 0{,}25 \left[2y_1(i) + y_2(i) + 4x_i^2 \right]$

$\Rightarrow y_1(i+1) = A + 0{,}25B$

$\Rightarrow y_2(i+1) = B + 0{,}25(2A + B + 4C^2)$

i	x_i	$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$		$y = y_1$ přesné ŘO
0	0 C	$\begin{pmatrix} 1 \\ 4 \end{pmatrix}$	A B	1
1	0,25 C	$\begin{pmatrix} 2 \\ 5{,}5 \end{pmatrix}$	D→A B	2,23
2	0,5 C	$\begin{pmatrix} 3{,}375 \\ 7{,}9375 \end{pmatrix}$	D→A B	4,1496
3	0,75 C	$\begin{pmatrix} 5{,}3594 \\ 11{,}8594 \end{pmatrix}$	D→A B	7,2831
4	1 C	$\begin{pmatrix} 8{,}3242 \\ 18{,}0664 \end{pmatrix}$	D→A B	12,5139
5	1,25 C	$\begin{pmatrix} 12{,}8408 \\ 27{,}7451 \end{pmatrix}$	D→A B	21,313
6	1,5 C	$\begin{pmatrix} 19{,}7771 \\ 42{,}6643 \end{pmatrix}$	D→A B	36,1173
7	1,75 C	$\begin{pmatrix} 30{,}4432 \\ 65{,}4689 \end{pmatrix}$	D→A B	60,9535
8	2	$\begin{pmatrix} 46{,}8104 \\ 100{,}1206 \end{pmatrix}$		102,467



Okrajové Problémy DR 2Ř

$$y'' = f(x; y; y') \quad ; y(a) = \alpha \; ; y(b) = \beta \; ;$$

→ odhad chybějící počáteční podmínky → Metoda střelby

→ převod na množinu algebraických rovnic → Metoda konečných diferencí

Metoda Střelby

$$\frac{d^2y}{dx^2} = y'' = f(x; y; y') \qquad y(a) = \alpha \qquad y(b) = \beta$$

+ odhadneme poč. podm. pro $y'(a) = u$ (zvolíme)

→ musíme najít správnou hodnotu u

→ u hledáme jako kořen rovnice → $y(b) = \varphi(u)$

→ u je kořen $r(u) = \varphi(u) - \beta = 0$ → $r(u)$ je okrajové reziduum

(rozdíl mezi skutečnou a vypočítanou okrajovou hodnotou)

Př 1: $y'' - 2y = 8x(9-x)$; $y(0) = 0$; $y(9) = 0$; $n = 3$; $x \in \langle 0; 9 \rangle$ Eulerova metoda

$$h = \frac{9-0}{3} = 3$$

$\alpha = 0$ $\beta = 0$

$x=0$ $x=3$ $x=6$ $x=9$

$y(0) = 0$

$y'(0) = 4$ ... odhaduji

a) Převod na soustavu ODR 1Ř

$(y = y_1)' \to y' = y_1' = y_2$

$(y' = y_2)' \to y'' = y_2'$

DDZ: $y_2' - 2 \cdot y_1 = 8x(9-x)$

$y_2' = 2y_1 + 8x(9-x)$

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= 2y_1 + 8x(9-x) \end{aligned}$$

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 8x(9-x) \end{pmatrix} \; ; \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \end{pmatrix}$$

b) IV:

$$y_{1(i+1)} = y_{1(i)} + h \cdot y_{2(i)}$$

$$y_{2(i+1)} = y_{2(i)} + h \cdot [2 \cdot y_{1(i)} + 8x_i(9-x_i)]$$

$$y_{1(i+1)} = y_{1(i)} + 3 \cdot y_{2(i)} \quad (= A + 3B$$

$$y_{2(i+1)} = y_{2(i)} + 6\, y_{1(i)} + 24 x_i (9 - x_i) \quad = B + 6A + 24C(9-C))$$

c)

i	x_i	$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$
0	0 → C	$\begin{pmatrix} 0 \\ 4 \end{pmatrix}$ → A, → B
1	3 → C	$\begin{pmatrix} 12 \\ 4 \end{pmatrix}$ → D → A, → B
2	6 → C	$\begin{pmatrix} 24 \\ 508 \end{pmatrix}$ → D → A, → B
3	9	$\begin{pmatrix} 1548 \\ 1084 \end{pmatrix}$

d) zvolil jsem $y'(0) = 4$ → $y(9) \approx 1548$ ale mělo být 0

proto zvolíme znovu $y'(0) = -24$

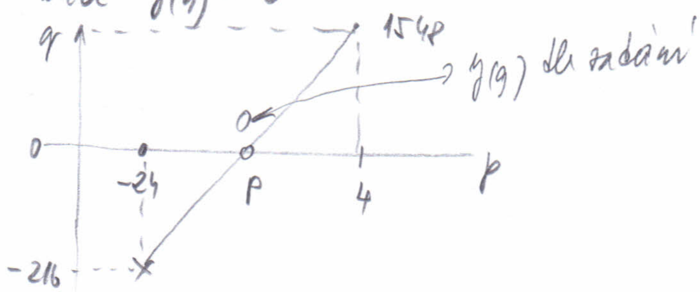
$$\begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} 0 \\ -24 \end{pmatrix}$$

i	x_i	$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$
0	0 → C	$\begin{pmatrix} 0 \\ -24 \end{pmatrix}$ → A, → B
1	3 → C	$\begin{pmatrix} -72 \\ -24 \end{pmatrix}$ → D → A, → B
2	6 → C	$\begin{pmatrix} -144 \\ -24 \end{pmatrix}$ → D → A, → B
3	9	$\begin{pmatrix} -216 \\ -456 \end{pmatrix}$

e) zvolil jsem $y'(0) = -24 \rightarrow y(9) \approx -216$

Interpolací zjistíme pro jaké $y'(0)$ bude $y(9) = 0$

$y'(0)$	$y(9)$
$p_0 = 4$	$1548 = q_0$
$p_1 = -24$	$-216 = q_1$



$$p = p_0 + \frac{p_1 - p_0}{q_1 - q_0}(q - q_0) \quad ; q_0 < q < q_1$$

$$p = 4 + \frac{-24-4}{-216-1548}(0 - 1548) \doteq -20{,}57$$

f) spočítali jsme, že $y'(0) = -20{,}57$

i	x_i	$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$	
0	$0 \rightarrow C$	$\begin{pmatrix} 0 \\ -20{,}57 \end{pmatrix}$	$\rightarrow A$ $\rightarrow B$
1	3^C	$\begin{pmatrix} -61{,}71 \\ -20{,}57 \end{pmatrix}$	$\rightarrow D \rightarrow A$ $\rightarrow B$
2	6^C	$\begin{pmatrix} -123{,}42 \\ 41{,}17 \end{pmatrix}$	$\rightarrow D \rightarrow A$ $\rightarrow B$
3	9	$\begin{pmatrix} 909 \\ -267{,}35 \end{pmatrix}$	

$$\begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} 0 \\ -20{,}57 \end{pmatrix}$$

$\rightarrow y(9) = 0{,}09 \approx 0$

Př 2 $y'' - 4x^2 y = -2e^{-x^2}$; $x \in \langle 0;1 \rangle$; $y(0) = 1$; $y(1) = e^{-1}$; $n = 4$

$h = \frac{1-0}{4} = 0{,}25$

a) $(y_1 = y)' \Rightarrow y' = y_1' = y_2$ DDZ: $y_2' - 4x^2 y_1 = -2e^{-x^2}$

$(y' = y_2)' \Rightarrow y'' = y_2'$

$$y_2' = 4x^2 y_1 - 2e^{-x^2}$$

maticově: $\left.\begin{array}{l} y_1' = \quad y_2 \\ y_2' = 4x^2 y_1 \quad -2e^{-x^2} \end{array}\right\}$ $\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 4x^2 & 0 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ -2e^{-x^2} \end{pmatrix}$; $y_1(0) = 1$

$y_1(1) = e^{-1}$

volím $y'(0) = y_2(0) = 4$

b) IV: $y_1(i+1) = y_1(i) + 0{,}25\,[y_2(i)]$

$$y_2(i+1) = y_2(i) + 0{,}25\left[4x_i^2 \cdot y_1(i) - 2e^{-x_i^2}\right]$$

paměti: $y_1(i+1) = A + 0{,}25 \cdot B$

$$y_2(i+1) = B + 0{,}25\left(4C^2 \cdot A - 2e^{-C^2}\right)$$

c)

i	x_i	$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$	
0	0^c	$\begin{pmatrix} 1 \\ 4 \end{pmatrix}$	$\to A$, $\to B$
1	$0{,}25^c$	$\begin{pmatrix} 2 \\ 3{,}5 \end{pmatrix}$	$\to D \to A$, $\to B$
2	$0{,}5^c$	$\begin{pmatrix} 2{,}875 \\ 3{,}1553 \end{pmatrix}$	$\to D \to A$, $\to B$
3	$0{,}75^c$	$\begin{pmatrix} 3{,}6638 \\ 3{,}4846 \end{pmatrix}$	$\to D \to A$, $\to B$
4	1^c	$\begin{pmatrix} 4{,}535 \\ 5{,}2607 \end{pmatrix}$	

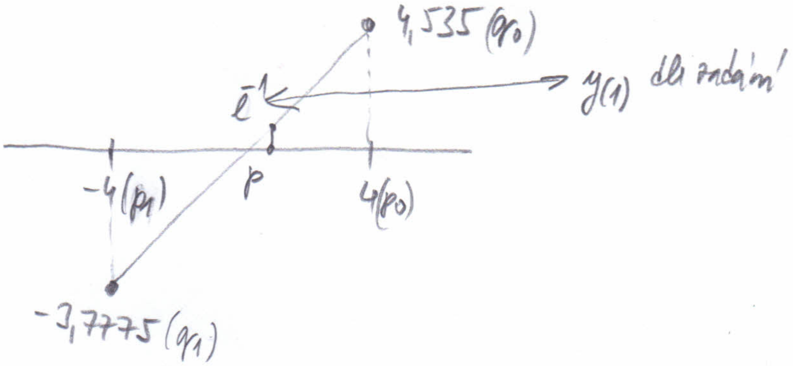
$y'(0) = 4 \Rightarrow y(1) \approx 4{,}535$

d)

volím $y'(0) = -4 \Rightarrow y(1) \approx -3{,}7775$

i	x_i	$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$	
0	0^c	$\begin{pmatrix} 1 \\ -4 \end{pmatrix}$	A, B
1	$0{,}25^c$	$\begin{pmatrix} 0 \\ -4{,}5 \end{pmatrix}$	$D \to A$, B
2	$0{,}5^c$	$\begin{pmatrix} -1{,}125 \\ -4{,}9697 \end{pmatrix}$	$D \to A$, B
3	$0{,}75^c$	$\begin{pmatrix} -2{,}3674 \\ -5{,}6404 \end{pmatrix}$	D, B
4	1^c	$\begin{pmatrix} -3{,}7775 \\ -7{,}2569 \end{pmatrix}$	

e) Interpolace



$p_0 = 4 \qquad q_0 = 4{,}535$

$p_1 = -4 \qquad q_1 = -3{,}7775$

$$p = p_0 + \frac{p_1 - p_0}{q_1 - q_0}\left(\overset{e^{-1}}{q} - q_0\right) \quad ; q_0 \le q \le q_1$$

$$p = 4 + \frac{-4-4}{-3{,}7775 - 4{,}535}\left(e^{-1} - 4{,}535\right) \doteq -0{,}0105$$

f)

i	x_i	$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$	
0	0^c	$\begin{pmatrix} 1 \\ -0{,}0105 \end{pmatrix}$	A, B
1	$0{,}25^c$	$\begin{pmatrix} 0{,}9974 \\ -0{,}5105 \end{pmatrix}$	$D \to A$, B
2	$0{,}5^c$	$\begin{pmatrix} 0{,}8698 \\ -0{,}9179 \end{pmatrix}$	$D \to A$, B
3	$0{,}75^c$	$\begin{pmatrix} 0{,}6403 \\ -1{,}0898 \end{pmatrix}$	$D \to A$, B
4	1	$\begin{pmatrix} 0{,}3678 \\ -1{,}0146 \end{pmatrix}$	

$y_2(0) = y_1'(0) = -0{,}0105$

$\to y_1(1) \doteq 0{,}3678 \doteq e^{-1}$