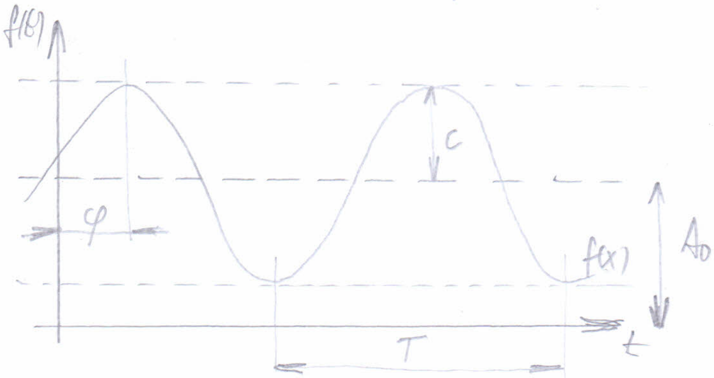
APROXIMACE TRIGONOMETRICKÝMI POLYNOMY

- data, která vykazují periodicitu (opakování) ⇒ oscilující systémy, vibrace
- použitím většího počtu trigonometrické řady ⇒ aproximace každé spojité fce na uzavřeném intervalu

→ limitní případ → konstrukce Fourierovy řady

→ periodická fce $\quad f(t) = f(t+T) \quad$ ... T perioda (nejmenší ze všech možných)



A_0 ... průměrná výška nad osou t

C ... amplituda

φ ... fázový posun (horizontální - osa t)

ω_0 ... úhlová frekvence

$$f(t) = A_0 + C\cdot\sin(\omega_0 t + \varphi) \quad \rightarrow \text{substituce } x = \omega_0 t$$

$$f(x) = A_0 + C\cdot\sin(x+\varphi) \quad \rightarrow \sin(x+\varphi) = \sin x\cos\varphi + \cos x\sin\varphi$$

$$f(x) = A_0 + C\cdot\sin x\cdot\cos\varphi + C\cos x\sin\varphi \quad \rightarrow A_1 = C\cdot\sin\varphi\ ;\ B_1 = C\cdot\cos\varphi$$

$$f(x) = A_0 + A_1\cos x + B_1\sin x + r \quad \rightarrow r \text{ je residuum}$$

(odchylka teoretické funkce a její aproximace)

předpokládáme body $[x_i; y_i]$; $i = 0,1,\dots,n$ $\quad$ MNČ budeme minimalizovat střední kvadratickou chybu $N[E_2(f)]^2$

$$N[E_2(f)]^2 = \sum_{i=0}^{N}\left[y_i - (A_0 + A_1\cos x_i + B_1\sin x_i)\right]^2$$

určujeme $(A_0\ A_1\ B_1)$

$$\frac{\partial N}{\partial A_0} = 2\sum_{i=0}^{N}\left\{\left[y_i - (A_0 + A_1\cos x_i + B_1\sin x_i)\right]\cdot(-1)\right\} = 0$$

(-1 je derivace z $-A_0$)

$$\frac{\partial N}{\partial A_1} = 2\sum_{i=0}^{N}\left\{\left[y_i - (A_0 + A_1\cos x_i + B_1\sin x_i)\right]\cdot(-\cos x_i)\right\} = 0$$

($-\cos x_i$ je derivace z $-A_1\cos x_i$ podle A_1)

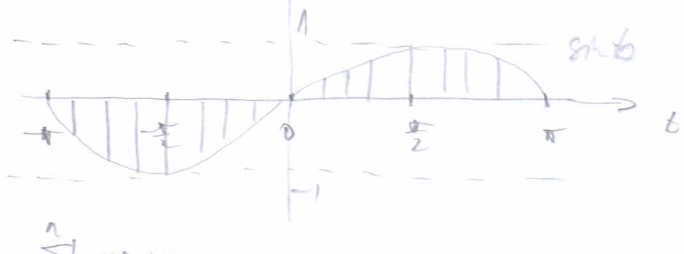
$$\frac{\partial N}{\partial B_1} = 2\sum_{i=0}^{M}\left\{\left[y_i - (A_0 + A_1\cos x_i + B_1\sin x_i)\right]\cdot(-\sin x_i)\right\} = 0$$

($-\sin x_i$ je derivace z $-B_1\sin x_i$ podle B_1)

NSR po úpravě:

$$\begin{bmatrix} N & \sum_{i=0}^{N}\cos x_i & \sum_{i=0}^{M}\sin x_i \\ \sum_{i=0}^{N}\cos x_i & \sum_{i=0}^{N}\cos^2 x_i & \sum_{i=0}^{N}\sin x_i\cos x_i \\ \sum_{i=0}^{N}\sin x_i & \sum_{i=0}^{N}\cos x_i\sin x_i & \sum_{i=0}^{M}\sin^2 x_i \end{bmatrix}\cdot\begin{bmatrix} A_0 \\ A_1 \\ B_1 \end{bmatrix} = \begin{bmatrix} \sum_{i=0}^{M} y_i \\ \sum_{i=0}^{N} y_i\cos x_i \\ \sum_{i=0}^{N} y_i\sin x_i \end{bmatrix}$$

pro ekvidistantní body x_i na intervalu $\langle -\pi; \pi \rangle$ platí



$$\sum_{i=0}^{N} \sin x_i = 0 \qquad \sum_{i=0}^{N} \cos x_i = 0$$

$$\sum_{i=0}^{N} \sin x_i \cos x_i = \frac{1}{2} \sum_{i=0}^{N} \sin 2x_i = 0$$

$$\sum_{i=0}^{N} \sin^2 x_i = \sum_{i=0}^{N} (1 - \cos^2 x_i) = N - \frac{N}{2} = \frac{N}{2} \qquad \sum_{i=0}^{N} \cos^2 x_i = \sum_{i=0}^{N} \frac{1+\cos 2x_i}{2} = \frac{N}{2} + \frac{1}{2} \underbrace{\sum_{i=0}^{N} \cos 2x_i}_{0} = \frac{N}{2}$$

NSR po dosazení
$$\underbrace{\begin{bmatrix} N & 0 & 0 \\ 0 & N/2 & 0 \\ 0 & 0 & N/2 \end{bmatrix}}_{\text{Regulární matice} \Rightarrow \exists \text{ matice inverzní}} \begin{bmatrix} A_0 \\ A_1 \\ B_1 \end{bmatrix} = \begin{bmatrix} \sum_{i=0}^{N} y_i \\ \sum_{i=0}^{N} y_i \cos x_i \\ \sum_{i=0}^{N} y_i \sin x_i \end{bmatrix}$$

$$\left(\begin{array}{ccc|ccc} N & 0 & 0 & 1 & 0 & 0 \\ 0 & N/2 & 0 & 0 & 1 & 0 \\ 0 & 0 & N/2 & 0 & 0 & 1 \end{array} \right) \begin{array}{l} :N \\ :N/2 \\ :N/2 \end{array} \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/N & 0 & 0 \\ 0 & 1 & 0 & 0 & 2/N & 0 \\ 0 & 0 & 1 & 0 & 0 & 2/N \end{array} \right) \Rightarrow IM \begin{pmatrix} 1/N & 0 & 0 \\ 0 & 2/N & 0 \\ 0 & 0 & 2/N \end{pmatrix}$$

$\Rightarrow$ vyjádříme koeficienty
$$\begin{bmatrix} A_0 \\ A_1 \\ B_1 \end{bmatrix} = \begin{bmatrix} 1/N & 0 & 0 \\ 0 & 2/N & 0 \\ 0 & 0 & 2/N \end{bmatrix} \begin{bmatrix} \sum_{i=0}^{N} y_i \\ \sum_{i=0}^{N} y_i \cos x_i \\ \sum_{i=0}^{N} y_i \sin x_i \end{bmatrix} \Rightarrow \begin{array}{l} A_0 = \frac{1}{N} \sum_{i=0}^{N} y_i \\ A_1 = 2/N \sum_{i=0}^{N} y_i \cos x_i \\ B_1 = 2/N \sum_{i=0}^{N} y_i \sin x_i \end{array}$$

ZOBECNĚNÍ: Periodickou fci s periodou 2π danou $n+1$ ekvidistantními body $[x_i; y_i]_{i=0}^{n}$ můžeme aproximovat TRIGONOMETRICKÝM POLYNOMEM (TP)

$$T_M(x) = A_0 + A_1 \cos x + B_1 \sin x + \dots + A_M \cos(Mx) + B_M \sin(Mx)$$

→ M je stupeň TP a platí pro něj: $N > 2M+1$

→ koeficienty spočítáme: $A_0 = 1/N \sum_{i=0}^{N} y_i$; $\left.\begin{array}{l} A_j = 2/N \sum_{i=0}^{N} y_i \cos(j x_i) \\ B_j = 2/N \sum_{i=0}^{N} y_i \sin(j x_i) \end{array}\right\} j = 1; \dots; M$

→ toto je numerická aproximace koeficientů Fourierovy řady

$$A_j = \int_{-\pi}^{\pi} f(x) \cos(jx)\, dx \rightarrow j = 0; 1; \dots; M$$

$$B_j = \int_{-\pi}^{\pi} f(x) \sin(jx)\, dx \rightarrow j = 1; \dots M$$

→ když perioda není 2π, ale T, musíme doplnit $\omega = \frac{2\pi}{T}$

$$[T_m(x) = A_0 + A_1 \cos(\omega x) + B_1 \sin(\omega x) + \dots]$$

Pr: Aproximujte fci danou tabulkou Trig. pol. 1. stupně

$T_1(x) = A_0 + A_1 \cos x + B_1 \sin x$

i	x_i	y_i	$y_i \cos x_i$	$y_i \sin x_i$
0	$-\pi$	2,2	-2,2	0
1	$-4\pi/5$	1,595	-1,29	-0,938
2	$-3\pi/5$	1,031	-0,319	-0,981
3	$-2\pi/5$	0,722	0,223	-0,687
4	$-\pi/5$	0,786	0,632	-0,462
5	0	1,2	1,2	0
6	$\pi/5$	1,805	1,46	1,061
7	$2\pi/5$	2,369	0,732	2,253
8	$3\pi/5$	2,678	-0,828	2,547
9	$4\pi/5$	2,614	-2,115	1,536
~~10~~	~~π~~	~~2,2~~	~~-2,2~~	~~0~~
	Σ	17	-2,501	4,329

$A_0 = \frac{17}{10} = 1,7$

$A_1 = \frac{2}{10}(-2,501) \doteq -0,5$

$B_1 = \frac{2}{10}(4,329) = 0,866$

$T_1(x) = 1,7 - 0,5 \cos x + 0,866 \sin x = 1,7 + \sin\left(x + 1,047 + \frac{\pi}{2}\right)$

převod na $y = A_0 + C \sin(\omega_0 t + \varphi)$ $\omega_0 = \cancel{2\pi}$ zvýší jako 1

$\varphi = \operatorname{arctg}\left(-\frac{B_1}{A_1}\right) = \operatorname{arctg}\left(-\frac{0,866}{-0,5}\right) \doteq 1,047$

$C = \sqrt{A_1^2 + B_1^2} = \sqrt{(-0,5)^2 + 0,866^2} \doteq 1$

V literatuře $T_M(x) = \frac{a_0}{2} + \sum_{j=1}^{M} a_j \cos(jx) + b_j \sin(jx)$

$a_j = \frac{2}{N} \sum_{k=0}^{N-1} y_k \cos(j x_k)$ $j = 0, 1, \dots, M$

$b_j = \frac{2}{N} \sum_{k=0}^{N-1} y_k \sin(j x_k)$ $j = 1, \dots, M$

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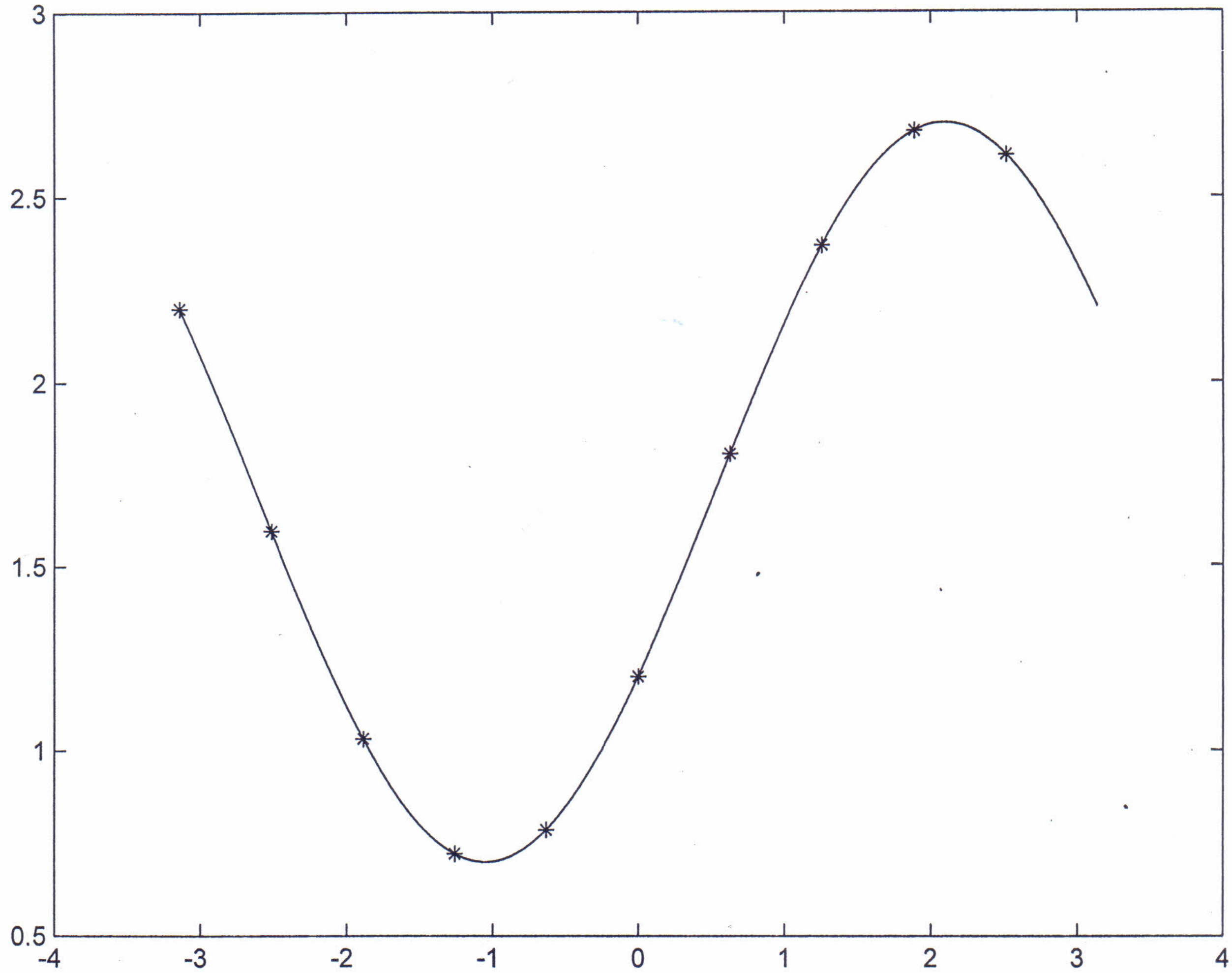


ad) Aproximujte trigonometrickým polynomem:

x_i	y_i	$y_i \cos \omega x_i$	$y_i \sin \omega x_i$	$y_i \cos 2\omega x_i$	$y_i \sin 2\omega x_i$	$y_i \cos 3\omega x_i$	$y_i \sin 3\omega x_i$
0	1	1	0	1	0	1	0
0,25	0,5	0,433	0,25	0,25	0,433	0	0,5
0,5	0	0	0	0	0	0	0
0,75	1	0	1	−1	0	0	−1
1	2	−1	1,7321	−1	−1,7321	2	0
1,25	0,5	−0,433	0,25	0,25	−0,433	0	0,5
1,5	−1	1	0	−1	0	1	0
1,75	0,5	−0,433	−0,25	0,25	0,433	0	−0,5
2	0	0	0	0	0	0	0
2,25	0,5	0	−0,5	−0,5	0	0	0,5
2,5	1	0,5	−0,866	−0,5	−0,866	−1	0
2,75	1,5	1,299	−0,75	0,75	−1,299	0	−1,5
Σ	7,5	2,366	0,866	−1,5	−3,4641	3	−1,5

$\omega = 2{,}0944 = \frac{2\pi}{3}$

$A_0 = \frac{1}{12} \cdot 7{,}5 = 0{,}625$ $\quad A_2 = \frac{2}{12} \cdot (-1{,}5) = -0{,}25$ $\quad A_3 = \frac{2}{12} \cdot 3 = 0{,}5$

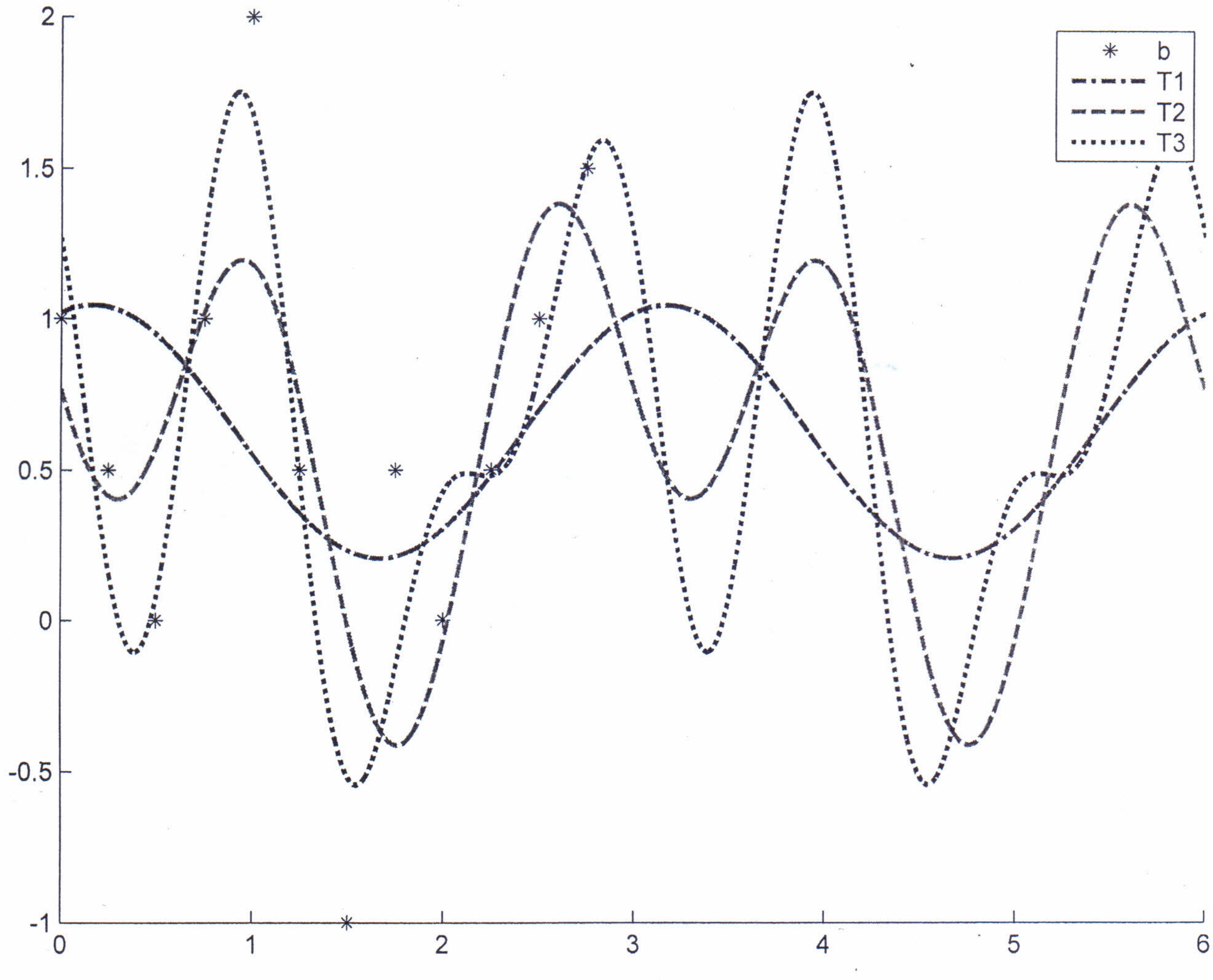
$A_1 = \frac{2}{12} \cdot 2{,}366 = 0{,}3943$ $\quad B_2 = \frac{2}{12} \cdot (-3{,}4641) = -0{,}5774$ $\quad B_3 = \frac{2}{12} \cdot (-1{,}5) = -0{,}25$

$B_1 = \frac{2}{12} \cdot 0{,}866 = 0{,}1443$

$$T_1(x) = 0{,}625 + 0{,}3943 \cos(2{,}0944x) + 0{,}1443 \sin(2{,}0944x)$$

$$T_2(x) = T_1(x) - 0{,}25 \cos(2 \cdot 2{,}0944x) - 0{,}5774 \sin(2 \cdot 2{,}0944x)$$

$$T_3(x) = T_2(x) + 0{,}5 \cos(3 \cdot 2{,}0944x) - 0{,}25 \sin(3 \cdot 2{,}0944x)$$


b
T1
T2
T3
2
1.5
1
0.5
0
-0.5
-1
0
1
2
3
4
5
6

Př: Aproximujte fci $f(x)$ danou tabulkou lineárním polynomem

a) pomocí NSR

b) pomocí ortogonalizace dle RALSTONA

i	0	1	2	3	4	5
x_i	-2	-1	0	1	2	3
y_i	-9	-2	-1	0	7	26

a) $P_1(x) = a_0\varphi_0(x) + a_1\varphi_1(x) = a_0 + a_1 x = \underline{\underline{\frac{3}{5} + \frac{29}{5}x}}$

$\varphi_0(x) = 1$

$\varphi_1(x) = x$

$a_0 = \frac{3}{5} \quad a_1 = \frac{29}{5}$

NSR: $a_0(1;1) + a_1(x;1) = (f;1)$

$a_0(1;x) + a_1(x;x) = (f;x)$

$$\xRightarrow{\text{dosazení}} \left(\begin{array}{cc|c} 6 & 3 & 21 \\ 3 & 19 & 112 \end{array}\right)^{:6,\,(-1)} \sim \left(\begin{array}{cc|c} 1 & 1/2 & 21/6 \\ 0 & 35/2 & 203/2 \end{array}\right)^{\cdot\frac{2}{35},\,(-1/2)} \sim \left(\begin{array}{cc|c} 1 & 0 & 3/5 \\ 0 & 1 & 29/5 \end{array}\right)$$

St.S: $(1;1) = \sum_{i=0}^{5}(1;1) = 1+1+1+1+1+1 = 6$

$(x;1) = (1;x) = \sum_{i=0}^{5}(x;1) = \sum_{i=0}^{5} x_i = \cancel{-2}\cancel{-1}+0\cancel{+1}\cancel{+2}+3 = 3$

$(x;x) = \sum_{i=0}^{5}(x;x) = \sum_{i=0}^{5} x_i^2 = 4+1+0+1+4+9 = 19$

$(f;1) = \sum_{i=0}^{5}(y_i;1) = \sum_{i=0}^{5} y_i = -9-2-1+0+7+26 = 21$

$(f;x) = \sum_{i=0}^{5}(y_i;x_i) \sum_{i=0}^{5} x_i y_i = 18+2+0+0+14+78 = 112$

b) $P_1(x) = \psi(x) = a_0\varphi_0(x) + a_1\varphi_1(x)$

$\varphi_{-1} = 0 \qquad b_0 = 0$

$\varphi_0 = 1$

$\varphi_1 = (x - c_0)\varphi_0 - b_0\varphi_{-1} = (x - c_0)\cdot 1 + 0 = x - \frac{1}{2}$

$$c_0 = \frac{(x\varphi_0;\varphi_0)}{(\varphi_0;\varphi_0)} = \frac{\sum_{i=0}^{5} x_i}{\sum_{i=0}^{5} 1} = \frac{3}{6} = \frac{1}{2}$$

$$a_0 = \frac{(f;\varphi_0)}{(\varphi_0;\varphi_0)} = \frac{\sum_{i=0}^{5} y_i}{\sum_{i=0}^{5} 1} = \frac{21}{6}$$

$$a_1 = \frac{(f;\varphi_1)}{(\varphi_1;\varphi_1)} = \frac{\sum_{i=0}^{5} y_i(x_i - \frac{1}{2})}{\sum_{i=0}^{5}(x_i - \frac{1}{2})^2} = \frac{-9(-2-\frac{1}{2}) - 2(-1-\frac{1}{2}) - 1(0-\frac{1}{2}) + 0(1-\frac{1}{2}) + 7(2-\frac{1}{2}) + 26(3-\frac{1}{2})}{(-2-\frac{1}{2})^2 + (-1-\frac{1}{2})^2 + (0-\frac{1}{2})^2 + (1-\frac{1}{2})^2 + (2-\frac{1}{2})^2 + (3-\frac{1}{2})^2} \ominus$$

$$\ominus \ \frac{\frac{203}{2}}{\frac{35}{2}} = \frac{203}{35} = \frac{29}{5}$$

$$\psi(x) = \frac{21}{6}\cdot 1 + \frac{29}{5}\left(x - \frac{1}{2}\right) = \underline{\underline{\frac{3}{5} + \frac{29}{5}x}}$$

Př: Aproximujte fci $f(x)$ danou tabulkou kvadratickým polynomem
a) pomocí NSR
b) pomocí ortogonalizace dle RALSTONA

i	0	1	2	3	4	5	6
x_i	-3	-2	-1	0	1	2	3
y_i	4	2	3	0	-1	-2	-5

a) $P_2(x) = a_0\varphi_0(x) + a_1\varphi_1(x) + a_2\varphi_2(x) = a_0 + a_1x + a_2x^2 = \frac{2}{3} - \frac{39}{28}x - \frac{11}{84}x^2$

$\varphi_0(x)=1 \quad \varphi_1(x)=x \quad \varphi_2(x)=x^2$

NSR
$$a_0(1;1) + a_1(x;1) + a_2(x^2;1) = (f;1)$$
$$a_0(1;x) + a_1(x;x) + a_2(x^2;x) = (f;x)$$
$$a_0(1;x^2) + a_1(x;x^2) + a_2(x^2;x^2) = (f;x^2)$$

Sk.S: $(1;1) = \sum_{i=0}^{6}(1;1) = 1+1+1+1+1+1+1 = 7$

$(x;1)=(1;x) = \sum_{i=0}^{6} x_i = -3-2-1+0+1+2+3 = 0$

$(x^2;1)=(x;x)=(1;x^2) = \sum_{i=0}^{6} x_i^2 = 9+4+1+0+1+4+9 = 28$

$(x^2;x)=(x;x^2) = \sum_{i=0}^{6} x_i^3 = -27-8-1+0+1+8+27 = 0$

$(x^2;x^2) = \sum_{i=0}^{6} x_i^4 = 81+16+1+0+1+16+81 = 196$

$(f;1) = \sum_{i=0}^{6} y_i = 4+2+3+0-1-2-5 = 1$

$(f;x) = \sum_{i=0}^{6} y_i x_i = -12-4-3+0-1-4-15 = -39$

$(f;x^2) = \sum_{i=0}^{6} y_i x_i^2 = 4\cdot9+2\cdot4+3\cdot1+0-1\cdot1-2\cdot4-5\cdot9$
$= -7$

$$\left(\begin{array}{ccc|c} 7 & 0 & 28 & 1 \\ 0 & 28 & 0 & -39 \\ 28 & 0 & 196 & -7 \end{array}\right) \begin{array}{l} :7 \\ :28 \\ \\ \end{array} \sim \left(\begin{array}{ccc|c} 1 & 0 & 4 & 1/7 \\ 0 & 1 & 0 & -39/28 \\ 0 & 0 & 84 & -11 \end{array}\right)_{\cdot(-28)\ :84} \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 2/3 \\ 0 & 1 & 0 & -39/28 \\ 0 & 0 & 1 & -11/84 \end{array}\right)_{\cdot(-4)}$$

$a_0 = 2/3$
$a_1 = -39/28$
$a_2 = -11/84$

b) $P_2(x) = \varphi(x) = a_0\varphi_0(x) + a_1\varphi_1(x) + a_2\varphi_2(x) \ominus$

$\varphi_0(x) = 1 \quad \varphi_{-1}(x) = 0 \quad b_0 = 0$

$\varphi_1(x) = (x-c_0)\varphi_0(x) - b_0\cdot\varphi_{-1}(x) = (x-0)\cdot 1 = x$

$\varphi_2(x) = (x-c_1)\varphi_1(x) - b_1\cdot\varphi_0(x) =$
$= (x-0)\cdot x - 4\cdot 1 = x^2-4$

$c_0 = \frac{(x\varphi_0;\varphi_0)}{(\varphi_0;\varphi_0)} = \frac{\sum_{i=0}^{6} x_i}{7} = \frac{0}{7} = 0$

$a_0 = \frac{(f;\varphi_0)}{(\varphi_0;\varphi_0)} = \frac{\sum_{i=0}^{6} y_i}{7} = \frac{1}{7}$

$c_1 = \frac{(x\varphi_1;\varphi_1)}{(\varphi_1;\varphi_1)} = \frac{\sum_{i=0}^{6} x_i^3}{\sum_{i=0}^{6} x_i^2} = \frac{0}{28} = 0$

$b_1 = \frac{(\varphi_1;\varphi_1)}{(\varphi_0;\varphi_0)} = \frac{\sum_{i=0}^{6} x_i^2}{7} = \frac{28}{7} = 4$

$a_1 = \frac{(f;\varphi_1)}{(\varphi_1;\varphi_1)} = \frac{\sum_{i=0}^{6} x_i y_i}{\sum_{i=0}^{6} x_i^2} = \frac{-39}{28}$

$a_2 = \frac{(f;\varphi_2)}{(\varphi_2;\varphi_2)} = \frac{\sum_{i=0}^{6} y_i(x_i^2-4)}{\sum_{i=0}^{6}(x_i^2-4)^2} = \frac{4(9-4)+2(4-4)+3(1-4)+0(0-4)-1(1-4)+}{(9-4)^2+(4-4)^2+(1-4)^2+(0-4)^2+(1-4)^2}$

$\frac{-2(4-4)-5(9-4)}{+(4-4)^2+(9-4)^2} = -\frac{11}{84}$

$\ominus\ \frac{1}{7} - \frac{39}{28}x - \frac{11}{84}(x^2-4) = \frac{2}{3} - \frac{39}{28}x - \frac{11}{84}x^2$

Př: MNČ aproximujte fci $y=\sqrt{x}$ na intervalu $\langle 0,1\rangle$ lineárním polynomem

a) pomocí NSR

b) pomocí ortogonalizace dle BALATONA

a) $y=\sqrt{x}$ $\qquad \varphi(x) = a_0\varphi_0(x) + a_1\varphi_1(x) = a_0 + a_1 x \ominus$

$\varphi_0(x)=1 \quad \varphi_1(x)=x$

NSR: $a_0(1;1) + a_1(1;x) = (f;1)$

$\underline{a_0(x;1) + a_1(x;x) = (f;x)}$

$$\left.\right\} \begin{pmatrix} 1 & 1/2 & 2/3 \\ 1/2 & 1/3 & 2/5 \end{pmatrix} \overset{\left(-\frac{1}{2}\right)}{\sim} \begin{pmatrix} 1 & 1/2 & 2/3 \\ 0 & 1/12 & 1/15 \end{pmatrix} \cdot 12 \sim$$

sk. s.: $(\varphi_0;\varphi_0) = (1;1) = \int_0^1 1\cdot 1\, dx = [x]_0^1 = 1$

$(\varphi_0;\varphi_1) = (1;x) = (x;1) = \int_0^1 x\cdot 1\, dx = \left[\frac{x^2}{2}\right]_0^1 = \frac{1}{2}$

$(\varphi_1;\varphi_1) = \int_0^1 x^2\, dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}$

$(f;\varphi_0) = \int_0^1 \sqrt{x}\, dx = \frac{2}{3}\left[x^{\frac{3}{2}}\right]_0^1 = \frac{2}{3}$

$(f;\varphi_1) = \int_0^1 x\sqrt{x}\, dx = \frac{2}{5}\left[x^{\frac{5}{2}}\right]_0^1 = \frac{2}{5}$

$$\sim \begin{pmatrix} 1 & 0 & 4/15 \\ 0 & 1 & 4/5 \end{pmatrix} \quad \left(-\frac{1}{2}\right) \qquad a_0 = \frac{4}{15} \quad a_1 = \frac{4}{5}$$

$\ominus\ \underline{\varphi(x) = \frac{4}{15} + \frac{4}{5}x}$

b) $y=\sqrt{x}$ $\qquad \varphi(x) = a_0\varphi_0(x) + a_1\varphi_1(x) \ominus$

$\varphi_{-1} = 0 \quad \varphi_0 = 1$

$\varphi_1 = (x-c_0)\varphi_0(x) - b_0\varphi_{-1}(x) = (x-c_0) = x - \frac{1}{2}$

$$c_0 = \frac{(x\varphi_0;\varphi_0)}{(\varphi_0;\varphi_0)} = \frac{\int_0^1 x\, dx}{\int_0^1 dx} = \frac{\frac{1}{2}}{1} = \frac{1}{2}$$

$$a_0 = \frac{(f;\varphi_0)}{(\varphi_0;\varphi_0)} = \frac{\int_0^1 \sqrt{x}\, dx}{\int_0^1 dx} = \frac{\frac{2}{3}}{1} = \frac{2}{3}$$

$$a_1 = \frac{(f;\varphi_1)}{(\varphi_1;\varphi_1)} = \frac{\int_0^1 \sqrt{x}\left(x-\frac{1}{2}\right) dx}{\int_0^1 \left(x-\frac{1}{2}\right)^2 dx} = \frac{\int_0^1 \left(x^{\frac{3}{2}} - \frac{1}{2}x^{\frac{1}{2}}\right) dx}{\int_0^1 \left(x^2 - x + \frac{1}{4}\right) dx} = \frac{\left[\frac{2}{5}x^{\frac{5}{2}} - \frac{1}{2}\cdot\frac{2}{3}\cdot x^{\frac{3}{2}}\right]_0^1}{\left[\frac{x^3}{3} - \frac{x^2}{2} + \frac{1}{4}x\right]_0^1} \ominus$$

$$\ominus\ \frac{\frac{2}{5} - \frac{1}{3}}{\frac{1}{3} - \frac{1}{2} + \frac{1}{4}} = \frac{4}{5}$$

$\ominus\ \underline{\varphi(x) = \frac{2}{3}\cdot 1 + \frac{4}{5}\left(x - \frac{1}{2}\right) = \frac{4}{15} + \frac{4}{5}x}$

Př. Aproximujte trigonometrickým polynomem

a) přes bázový systém $\{1; \sin x; \cos x\}$

b) trigonometrickou aproximací

x_i	$\pi/2$	π	$3\pi/2$	2π
f_i	$1/2$	1	$1/2$	0

$n > 2m+1$

$4 > 2\cdot 1+1 \qquad M = 0;1$

a) $\varphi(x) = a_0\varphi_0(x) + a_1\varphi_1(x) + a_2\varphi_2(x) = a_0 + a_1 \sin x + a_2 \cdot \cos x \ominus$

$\Rightarrow$ diagonální soustava: $a_0(\varphi_0;\varphi_0) = (f;\varphi_0)$

$a_1(\varphi_1;\varphi_1) = (f;\varphi_1)$

$a_2(\varphi_2;\varphi_2) = (f;\varphi_2)$

$a_0 = \frac{(f;\varphi_0)}{(\varphi_0;\varphi_0)} = \frac{2}{4} = \frac{1}{2}$

$a_1 = \frac{(f;\varphi_1)}{(\varphi_1;\varphi_1)} = \frac{0}{2} = 0$

$a_2 = \frac{(f;\varphi_2)}{(\varphi_2;\varphi_2)} = \frac{-1}{2} = -\frac{1}{2}$

$(\varphi_0;\varphi_0) = \sum_{i=0}^{3} 1 = 4$

$(\varphi_1;\varphi_1) = \sum_{i=0}^{3} \sin x_i \cdot \sin x_i = \sum_{i=0}^{3} \sin^2 x_i = 1^2 + 0^2 + 1^2 + 0^2 = 2$

$(\varphi_2;\varphi_2) = \sum_{i=0}^{3} \cos^2 x_i = (0^2 + (-1)^2 + 0^2 + 1^2) = 2$

$\ominus\ \varphi(x) = \frac{1}{2} - \frac{1}{2}\cos x = \underline{\frac{1}{2}(1-\cos x)}$

$(\varphi_0;f) = \sum_{i=0}^{3} f_i = \frac{1}{2} + 1 + \frac{1}{2} + 0 = 2$

$(\varphi_1;f) = \sum_{i=0}^{3} f_i \cdot \sin x_i = \frac{1}{2}\overset{1}{\sin\frac{\pi}{2}} + 1\cdot\overset{0}{\sin\pi} + \frac{1}{2}\cdot\overset{-1}{\sin\frac{3\pi}{2}} + 0\cdot\overset{0}{\sin 2\pi} = 0$

$(\varphi_2;f) = \sum_{i=0}^{3} f_i \cos x_i = \frac{1}{2}\overset{0}{\cos\frac{\pi}{2}} + 1\cdot\overset{-1}{\cos\pi} + \frac{1}{2}\overset{0}{\cos\frac{3\pi}{2}} + \cancel{0\cdot\cos 2\pi} = -1$

b) $\varphi(x) = a_0 + \sum_{k=1}^{M}(a_k \cdot \cos kx + b_k \sin kx) \ominus$

$n = 4 \qquad 4 > 2m+1 \qquad m = 0;1$

$a_0 = \frac{1}{n}\sum_{i=0}^{N} y_i$

$a_k = \frac{2}{N}\sum_{i=0}^{N} f_i \cos kx_i$

$b_k = \frac{2}{N}\sum_{i=0}^{N} f_i \sin kx_i$

$a_0 = \frac{1}{n}\sum_{i=0}^{N} y_i = \frac{1}{4}\left(\frac{1}{2} + 1 + \frac{1}{2} + 0\right) = \frac{1}{2}$

$a_1 = \frac{2}{N}\sum_{i=0}^{N} y_i \cos x_i = \frac{1}{2}\left(\frac{1}{2}\cos\frac{\pi}{2} + 1\cdot\cos\pi + \frac{1}{2}\cos\frac{3\pi}{2} + 0\cos 2\pi\right) = \frac{1}{2}(0 - 1 + 0 + 0) = -\frac{1}{2}$

$b_1 = \frac{2}{N}\sum_{i=0}^{M} f_i \sin x_i = \frac{1}{2}\left(\frac{1}{2}\sin\frac{\pi}{2} + 1\cdot\sin\pi + \frac{1}{2}\sin\frac{3\pi}{2} + 0\cdot\sin 2\pi\right) = \frac{1}{2}\left(\frac{1}{2} + 0 - \frac{1}{2} + 0\right) = 0$

$\ominus\ \varphi(x) = \frac{1}{2} + \left(-\frac{1}{2}\cos x + 0\cdot\sin x\right) = \frac{1}{2} - \frac{1}{2}\cos x = \underline{\frac{1}{2}(1-\cos x)}$

PW, 8 węzłów

x_i	$\pi/4$	$\pi/2$	$3\pi/4$	π	$5\pi/4$	$3/2\pi$	$7/4\pi$	2π
f_i	1/4	1/2	3/4	1	3/4	1/2	1/4	0

$n=8$

$n>2m+1$

$8>2m+1$

$m<\frac{7}{2}$ $m=0,1,2,3$

$m=3$ (interpolacja byłaby możliwa)

0. $m=0$

$$a_0=\frac{2}{n}\sum_{i=1}^{8} f_i\cdot\cos 0x_i=\frac{2}{8}\left(\frac{1}{4}\cdot\cos 0+\frac{1}{2}\cdot\cos 0+\frac{3}{4}\cdot\cos 0+1\cdot\cos 0+\frac{3}{4}\cos 0+\frac{1}{2}\cos 0+\frac{1}{4}\cdot\cos 0+0\cdot\cos 0\right)=$$
$$=\frac{1}{4}\left(\frac{1}{4}+\frac{1}{2}+\frac{3}{4}+1+\frac{3}{4}+\frac{1}{2}+\frac{1}{4}+0\right)=\frac{1}{4}\cdot 4=\underline{1}$$

1. $m=1$

$$a_1=\frac{2}{n}\sum_{i=1}^{8} f_i\cos x_i=\frac{1}{4}\left(\frac{1}{4}\cdot\cos\frac{\pi}{4}+\frac{1}{2}\cos\frac{\pi}{2}+\frac{3}{4}\cdot\cos\frac{3\pi}{4}+1\cdot\cos\pi+\frac{3}{4}\cos\frac{5\pi}{4}+\frac{1}{2}\cos\frac{3\pi}{2}+\frac{1}{4}\cos\frac{7\pi}{4}+0\cos 2\pi\right)=$$
$$=\frac{1}{4}\left(\frac{1}{4}\cdot\frac{\sqrt2}{2}+\cancel{\frac{1}{2}\cdot 0}+\frac{3}{4}\cdot\left(-\frac{\sqrt2}{2}\right)+1\cdot(-1)+\frac{3}{4}\left(-\frac{\sqrt2}{2}\right)+\cancel{\frac{1}{2}\cdot 0}+\frac{1}{4}\left(+\frac{\sqrt2}{2}\right)+0\right)=$$
$$=\frac{1}{4}\left(\frac{\sqrt2}{8}-\frac{3\sqrt2}{8}-1-\frac{3\sqrt2}{8}+\frac{\sqrt2}{8}\right)=\frac{1}{4}\left(-\frac{4\sqrt2}{8}-1\right)=\underline{-\frac{\sqrt2}{8}-\frac{1}{4}}$$

$$b_1=\frac{2}{n}\sum_{i=1}^{8} f_i\sin x_i=\frac{1}{4}\left(\frac{1}{4}\sin\frac{\pi}{4}+\frac{1}{2}\sin\frac{\pi}{2}+\frac{3}{4}\sin\frac{3\pi}{4}+1\cdot\sin\pi+\frac{3}{4}\sin\frac{5\pi}{4}+\frac{1}{2}\sin\frac{3\pi}{2}+\frac{1}{4}\sin\frac{7\pi}{4}+0\sin 2\pi\right)=$$
$$=\frac{1}{4}\left(\frac{1}{4}\cdot\frac{\sqrt2}{2}+\frac{1}{2}\cdot 1+\frac{3}{4}\cdot\frac{\sqrt2}{2}+0+\frac{3}{4}\left(-\frac{\sqrt2}{2}\right)+\frac{1}{2}(-1)+\frac{1}{4}\left(-\frac{\sqrt2}{2}\right)+0\right)=$$
$$=\frac{1}{4}\left(\cancel{\frac{\sqrt2}{8}}+\cancel{\frac{1}{2}}+\cancel{\frac{3\sqrt2}{8}}-\cancel{\frac{3\sqrt2}{8}}-\cancel{\frac{1}{2}}-\cancel{\frac{\sqrt2}{8}}\right)=\frac{1}{4}\cdot 0=0$$

$$\varphi_1(x)=\frac{1}{2}+\left(-\frac{\sqrt2}{8}-\frac{1}{4}\right)\cdot\cos x$$

2. $m=2$ a_0 i a_1, b_1 mamy, policzmy a_2, b_2

$$a_2=\frac{2}{n}\sum_{i=1}^{8} f_i\cdot\cos 2x_i=\frac{1}{4}\left(\frac{1}{4}\cdot\cos\frac{\pi}{2}+\frac{1}{2}\cdot\cos\pi+\frac{3}{4}\cdot\cos\frac{3\pi}{2}+1\cdot\cos 2\pi+\frac{3}{4}\cos\frac{5\pi}{2}+\frac{1}{2}\cos 3\pi+\frac{1}{4}\cdot\cos\frac{7}{2}\pi+0\cdot\cos 4\pi\right)=$$
$$=\frac{1}{4}\left(\cancel{\frac{1}{4}\cdot 0}+\frac{1}{2}\cdot(-1)+\cancel{\frac{3}{4}\cdot 0}+1+\cancel{\frac{3}{4}\cdot 0}+\frac{1}{2}(-1)+\cancel{\frac{1}{4}\cdot 0}+0\right)=\frac{1}{4}\left(-\frac{1}{2}+1-\frac{1}{2}\right)=\frac{1}{4}\cdot 0=\underline{0}$$

$$b_2=\frac{2}{n}\sum_{i=1}^{8} f_i\sin 2x_i=\frac{1}{4}\left(\frac{1}{4}\cdot\sin\frac{\pi}{2}+\frac{1}{2}\sin\pi+\frac{3}{4}\sin\frac{3\pi}{2}+1\sin 2\pi+\frac{3}{4}\sin\frac{5\pi}{2}+\frac{1}{2}\sin 3\pi+\frac{1}{4}\sin\frac{7\pi}{2}+0\cdot\sin 4\pi\right)=$$
$$=\frac{1}{4}\left(\cancel{\frac{1}{4}}+0-\cancel{\frac{3}{4}}+0+\cancel{\frac{3}{4}}+0-\cancel{\frac{1}{4}}+0\right)=\frac{1}{4}\cdot 0=\underline{0}$$

$$\varphi_2(x)=\varphi_1(x)=\frac{1}{2}+\left(-\frac{\sqrt2}{8}-\frac{1}{4}\right)\cos x$$

3. $m=3$ a_0-a_2, b_1, b_2 mamy, policzmy a_3, b_3

$$a_3=\frac{2}{n}\sum_{i=1}^{8} f_i\cos 3x_i=\frac{1}{4}\left(\frac{1}{4}\cos\frac{3\pi}{4}+\frac{1}{2}\cos\frac{3\pi}{2}+\frac{3}{4}\cos\frac{9\pi}{4}+1\cdot\cos 3\pi+\frac{3}{4}\cdot\cos\frac{15\pi}{4}+\frac{1}{2}\cdot\cos\frac{9\pi}{2}+\frac{1}{4}\cos\frac{21}{4}\pi+0\cos 6\pi\right)=$$
$$=\frac{1}{4}\left(\frac{1}{4}\cdot\left(-\frac{\sqrt2}{2}\right)+\cancel{\frac{1}{2}\cdot 0}+\frac{3}{4}\cdot\frac{\sqrt2}{2}-1+\frac{3}{4}\left(\frac{\sqrt2}{2}\right)+0+\frac{1}{4}\left(-\frac{\sqrt2}{2}\right)+0\right)=$$
$$=\frac{1}{4}\left(-\frac{\sqrt2}{8}+\frac{3\sqrt2}{8}-\frac{8}{8}+\frac{3\sqrt2}{8}+0-\frac{\sqrt2}{8}+0\right)=\frac{1}{4}\left(\frac{4\sqrt2}{8}-1\right)=\underline{\frac{\sqrt2}{8}-\frac{1}{4}}$$

$$b_3=\frac{2}{n}\sum_{i=1}^{8} f_i\sin 3x_i=\frac{1}{4}\left(\frac{1}{4}\sin\frac{3\pi}{4}+\frac{1}{2}\sin\frac{3\pi}{2}+\frac{3}{4}\sin\frac{9\pi}{4}+1\sin 3\pi+\frac{3}{4}\sin\frac{15\pi}{4}+\frac{1}{2}\sin\frac{9\pi}{2}+\frac{1}{4}\sin\frac{21\pi}{4}+0\sin 6\pi\right)=$$
$$=\frac{1}{4}\left(\cancel{\frac{1}{4}\frac{\sqrt2}{2}}-\cancel{\frac{1}{2}}+\cancel{\frac{3}{4}\frac{\sqrt2}{2}}+0-\cancel{\frac{3}{4}\frac{\sqrt2}{2}}+\cancel{\frac{1}{2}}-\cancel{\frac{1}{4}\cdot\frac{\sqrt2}{2}}+0\right)=\frac{1}{4}(0)=\underline{0}$$

$$\underline{\varphi_3(x)=\varphi(x)=\frac{1}{2}+\left(-\frac{\sqrt2}{8}-\frac{1}{4}\right)\cos x+\left(\frac{\sqrt2}{8}-\frac{1}{4}\right)\cos 3x}$$