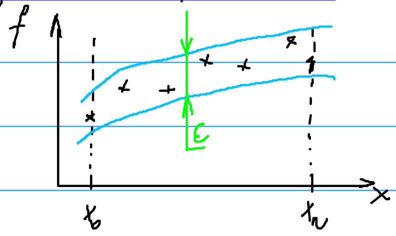
Aproximace Funkcí

$[x_i; f(x_i)]$	Póly
x_0	f_0
x_1	f_1
$\vdots$	
x_N	f_N

→ nahrazení složitých fcí (vhodná jednodušší fce)

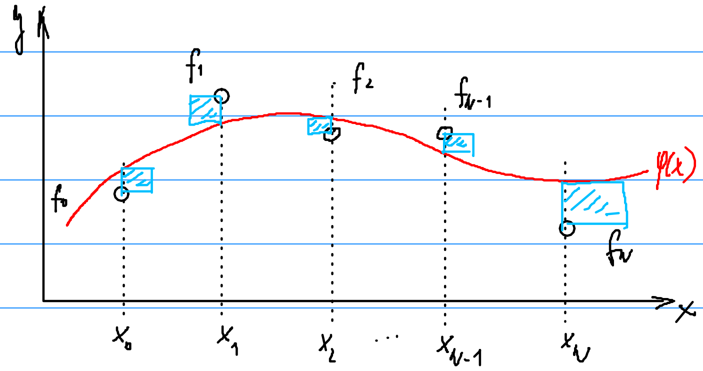
Typy: 1) <u>Interpolace</u> $f(x) \sim \varphi(x)$ $\quad \varphi(x_i) = f_i;\ \forall x_i;\ i=0,\dots,N$

2) <u>Stejnoměrná aproximace</u> (není jednoznačná)



$\forall x_i \in \langle x_0; x_N \rangle \qquad \max |f(x) - \varphi(x)| \le \varepsilon$

3) Metoda nejmenších čtverců



INTERPOLACE

obecná formulace úlohy:

VSTUP: n+1 hodnot

x_0	f_0
x_1	f_1
$\vdots$	
x_N	f_N

$[x_i; f_i]$; $i = 0, 1, \dots, N$ PÓLY

VÝSTUP: $\varphi(x) \longrightarrow$ tvar fce

→ • polynomické PI

→ • splajnová

→ • trigonometrické

→ . . . .

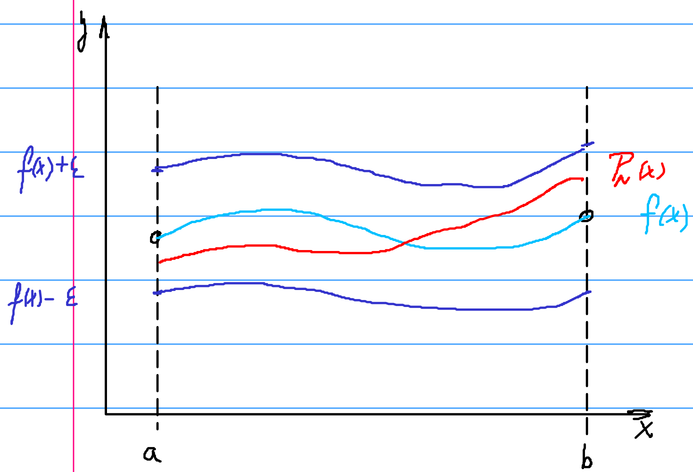
→ kombinované

$a_i;\ i=0,1,\dots,N$ koeficienty

$\varphi(x)$ ... třídy aproximačních fcí $\quad \varphi(\underbrace{x}_{\text{proměnná}}; a_0; a_1; \dots; a_N)$

PI: $\varphi(x) = P_N(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_N x^N$ ... (snadno D, $\int$)

WEIERSTRASSOVA VĚTA O APROXIMACI



Je-li $f(x)$ spojitá fce definovaná na I; $x \in \langle a; b \rangle$ a $\varepsilon > 0$.

Pak $\exists!$ polynom $P_n(x)$ definovaný na $\langle a; b \rangle$ takový, že platí: $|f(x) - P_n(x)| < \varepsilon;\ \forall x \in \langle a; b \rangle$

$[x_i; y_i]$; $i = 0, 1, \dots N$ $\xrightarrow{(N+1)}$ $x_i \neq x_j$ (žádné 2 x-ové hodnoty nejsou stejné)

→ ∃! polynom $P_n(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_N x^N$ → nejvýše n-tého stupně (N)

→ $P_N(x_i) = y_i$; $\forall i = 0; 1; \dots; N$

$$P_N(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^N$$

$$\left.\begin{aligned}
[x_0; y_0] \in P_N(x):&\quad y_0 = a_0 + a_1 x_0 + a_2 x_0^2 + \dots + a_n x_0^N \\
[x_1; y_1] \in P_N(x):&\quad y_1 = a_0 + a_1 x_1 + a_2 x_1^2 + \dots + a_n x_1^N \\
&\quad \dots \\
[x_N; y_N] \in P_N(x):&\quad y_N = a_0 + a_1 x_N + a_2 x_N^2 + \dots + a_n x_N^N
\end{aligned}\right\}$$

N+1 rovnic o N+1 neznámých $(a_0; a_1; \dots; a_N)$

∃! řešení ? $x_0 < x_1 < x_2 < \dots < x_N$?

$$D = \begin{vmatrix} 1 & x_0 & x_0^2 & \dots & x_0^N \\ 1 & x_1 & x_1^2 & \dots & x_1^N \\ \dots & & & & \\ 1 & x_N & x_N^2 & \dots & x_N^N \end{vmatrix} = (x_2 - x_1)(x_3 - x_2) \cdots (x_N - x_{N-1}) \neq 0 \Rightarrow \exists! \text{ řešení}$$

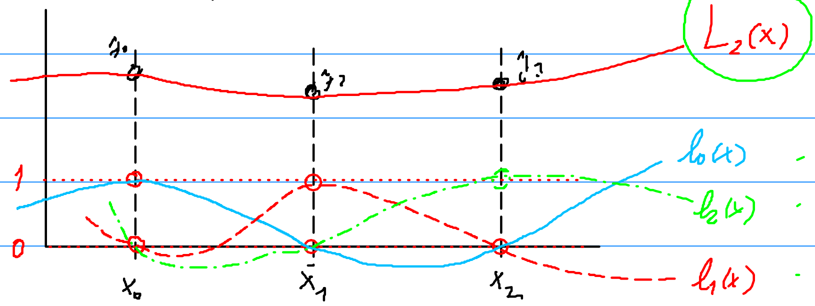
Vandermondův D

MNK → KRAMER špatně podmíněné metody → velkou výpočetní náročnost

LAGRANGEUV IP

→ nepřímá konstrukce pomocí fundamentálních plí. $l_i(x)$; $i = 0, 1, \dots, N$

→ $l_i(x_j) = \begin{cases} 0 & ; i \neq j \\ 1 & ; i = j \end{cases}$



→ $l_i(x)$ mají kořeny $x_0, x_1, \dots; x_{i-1}; x_{i+1}; \dots x_N$

$$l_i(x) = C_i (x - x_0)(x - x_1) \dots (x - x_{i-1})(x - x_{i+1}) \dots (x - x_N)$$

$$l_i(x_i) = 1 \rightarrow C_i = \frac{1}{(x_i - x_0)(x_i - x_1) \dots (x_i - x_{i-1})(x_i - x_{i+1}) \dots (x_i - x_N)}$$

$$l_i(x) = \frac{(x - x_0)(x - x_1) \dots (x - x_{i-1})(x - x_{i+1}) \dots (x - x_N)}{(x_i - x_0)(x_i - x_1) \dots (x_i - x_{i-1})(x_i - x_{i+1}) \dots (x_i - x_N)}$$

$$L_N(x) = \sum_{i=0}^{N} y_i \, l_i(x)$$

Př 1:

x_i	0	1	2	5
y_i	2	3	12	147

(x^2-3x+2)

FP:

$$l_0(x) = \frac{(x-1)(x-2)(x-5)}{\underbrace{(0-1)}_{-1}\underbrace{(0-2)}_{-2}\underbrace{(0-5)}_{-5}} = -\frac{1}{10}\left(x^3-8x^2+17x-10\right)$$

$$l_1(x) = \frac{(x-0)(x-2)(x-5)}{\underbrace{(1-0)}_{1}\underbrace{(1-2)}_{-1}\underbrace{(1-5)}_{-4}} = \frac{1}{4}\left(x^3-7x^2+10x\right)$$

$$l_2(x) = \frac{(x-0)(x-1)(x-5)}{\underbrace{(2-0)}_{2}\underbrace{(2-1)}_{1}\underbrace{(2-5)}_{-3}} = -\frac{1}{6}\left(x^3-6x^2+5x\right)$$

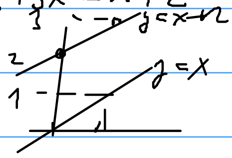
$$l_3(x) = \frac{(x-0)(x-1)(x-2)}{\underbrace{(5-0)}_{5}\underbrace{(5-1)}_{4}\underbrace{(5-2)}_{3}} = \frac{1}{60}\left(x^3-3x^2+2x\right)$$

$$L_3(x) = 2\cdot\left(-\frac{1}{10}\right)\left(x^3-8x^2+17x-10\right) + 3\cdot\frac{1}{4}\left(x^3-7x^2+10x\right) + 12\left(-\frac{1}{6}\right)\left(x^3-3x^2+2x\right) +$$
$$+ 147\left(\frac{1}{60}\right)\left(x^3-3x^2+2x\right) = \underline{\underline{x^3+x^2-x+2}}$$

Př 2: přímka 2 body

x_i	0	1
y_i	2	3

$$p:\ 2\cdot\frac{x-1}{\underbrace{0-1}_{-1}} + 3\cdot\frac{x-0}{\underbrace{1-0}_{1}} = -2x+2+3x = x+2$$



Newtonův IP (NIP)

LIP … teoretický význam

Poměrné diference

x_i	f_i	$\Delta^1 f(x)$	$\Delta^2 f(x)$	…	$\Delta^{N-1} f(x)$	$\Delta^N f(x)$
x_0	f_0					
x_1	f_1	$f[x_0;x_1]$				
x_2	f_2	$f[x_1;x_2]$	$f[x_0;x_1;x_2]$			
x_3	f_3	$f[x_2;x_3]$	$f[x_1;x_2;x_3]$			
x_4	f_4	$f[x_3;x_4]$	$f[x_2;x_3;x_4]$			
…	…	…	…			
x_{N-1}	f_{N-1}	$f[x_{N-2};x_{N-1}]$	$f[x_{N-3};x_{N-2};x_{N-1}]$		$f[x_0;\dots;x_{N-1}]$	
x_N	f_N	$f[x_{N-1};x_N]$	$f[x_{N-2};x_{N-1};x_N]$		$f[x_1;\dots;x_N]$	$f[x_0;\dots;x_N]$

$$\Delta^0 f(x) = f_i$$

$$\Delta^1 f(x) = f[x_0;x_1] = \frac{f_1 - f_0}{x_1 - x_0} \qquad f[x_1;x_2] = \frac{f_2 - f_1}{x_2 - x_1} \qquad \dots \qquad f[x_{N-1};x_N] = \frac{f_N - f_{N-1}}{x_N - x_{N-1}}$$

$$\Delta^2 f(x) = f[x_0;x_1;x_2] = \frac{f[x_1;x_2] - f[x_0;x_1]}{x_2 - x_0} \qquad \dots \qquad f[x_{N-2};x_{N-1};x_N] = \frac{f[x_{N-1};x_N] - f[x_{N-2};x_{N-1}]}{x_N - x_{N-2}}$$

…

$$\Delta^N f(x) = f[x_0;\dots;x_N] = \frac{f[x_1;\dots;x_N] - f[x_0;\dots;x_{N-1}]}{x_N - x_0}$$

NIP $\quad N_N(x) = f(x_0) + f[x_0;x_1](x-x_0) + f[x_0;x_1;x_2](x-x_0)(x-x_1) + \dots$
$\quad + \dots f[x_0;\dots;x_N](x-x_0)(x-x_1)\dots(x-x_{N-1})$

Př 3:

x_i	y_i	$\Delta^1 f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
0	2			
1	3	$\frac{3-2}{1-0}$ 1		
2	12	$\frac{12-3}{2-1}$ 9	$\frac{9-1}{2-0}$ 4	
5	147	$\frac{147-12}{5-2}$ 45	$\frac{45-9}{5-1}$ 9	$\frac{9-4}{5-0}$ 1

$$N_3(x) = \overset{2}{2} + \overset{x}{1\cdot(x-0)} + \overset{4x^2-4x}{4\cdot(x-0)(x-1)} +$$

$$+ 1(x-0)\overset{x^2-3x+2}{(x-1)(x-2)} =$$
x^3-3x^2+2x

$$= \underline{\underline{2 - x + x^2 + x^3}}$$

Př 4:

x_i	y_i	$\Delta^1 f(x)$
0	2	
1	3	$\frac{3-2}{1-0}$ 1

$$N_1(x) = 2 + 1\cdot(x-0) = 2 + x$$

Odhad Chyby LIP a NIP

→ $f(x)$

→ $P_N(x)$

$$f(x) = P_N(x) + \frac{f^{(N+1)}(\xi)}{(N+1)!}(x-x_0)(x-x_1)\dots(x-x_N)$$

$$f(x) - P_N(x) = \frac{f^{(N+1)}(\xi)}{(N+1)!}(x-x_0)(x-x_1)\dots(x-x_N)$$

$$|f^{(N+1)}(\xi)| \le M \quad \text{pro } x \in \langle x_0; x_N \rangle$$

↳ hledáme max hodnotu $(N+1)$. derivace na $x \in \langle x_0; x_N \rangle$

Př 5

x	0	$\pi/6$	$\pi/4$	$\pi/3$
$\sin x$	0	$1/2$	$\sqrt{2}/2$	$\sqrt{3}/2$

$\Rightarrow L_3(x)$

$(3+1). \to 4.$ derivace $\sin x$

$(\sin x)' \Rightarrow (\cos x)' \Rightarrow (-\sin x)' = (-\cos x)' \to \sin x$

M odhad z $\max \sin x$ $x \in \langle 0; \frac{\pi}{3} \rangle$

$M = 1$

$x = \frac{\pi}{5}$ $\quad L_3\left(\frac{\pi}{5}\right) \approx \sin\frac{\pi}{5}$

$$\left|\sin\frac{\pi}{5} - L_3\left(\frac{\pi}{5}\right)\right| \le \underbrace{\frac{1}{4!}\left(\frac{\pi}{5}-0\right)\left(\frac{\pi}{5}-\frac{\pi}{6}\right)\left|\frac{\pi}{5}-\frac{\pi}{4}\right|\left|\frac{\pi}{5}-\frac{\pi}{3}\right)}_{\underline{\underline{0{,}180387\cdot 10^{-3}}}}$$