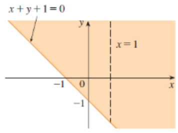
XAMAT, YAMAT

Cvičení 9

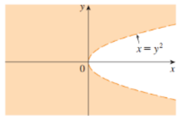
Diferenciální a integrální počet funkcí dvou a více reálných proměnných

1. Vypočtěte pro každou z uvedených funkcí $f(-2;2)$ a načrtněte jejich definiční obor.

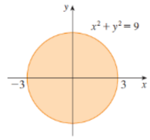
$a)\ f(x;y) = \frac{\sqrt{x+y+1}}{x-1}$ $\left\{-\frac{1}{3}\right\}$



$b)\ f(x;y) = x\ln(y^2 - x)$ $\{-2\ln 6\}$



$c)\ f(x;y) = \sqrt{9 - x^2 - y^2}$ $\{1\}$



2. Určete obě parciální derivace funkcí

$a)\ f(x;y) = \frac{x^2 y}{x^2 + y^2}$ $\left\{\frac{\partial f}{\partial x} = \frac{2xy^3}{(x^2+y^2)^2}; \frac{\partial f}{\partial y} = \frac{x^2(x^2-y^2)}{(x^2+y^2)^2}\right\}$

$a)\ f(x;y) = \frac{x}{\sqrt{x^2+y^2}}$ $\left\{\frac{\partial f}{\partial x} = \frac{y^2}{(x^2+y^2)^{\frac{3}{2}}}; \frac{\partial f}{\partial y} = \frac{-xy}{(x^2+y^2)^{\frac{3}{2}}}\right\}$

3. Určete všechny první i druhé parciální derivace funkcí

$a)\ f(x;y) = \ln(2x+3y)$

$$\left\{\begin{array}{c} \frac{\partial f}{\partial x} = \frac{2}{2x+3y}; \frac{\partial f}{\partial y} = \frac{3}{2x+3y}; \\ \frac{\partial^2 f}{\partial x^2} = \frac{-4}{(2x+3y)^2}; \frac{\partial^2 f}{\partial y^2} = \frac{-9}{(2x+3y)^2}; \frac{\partial^2 f}{\partial x \partial y} = \frac{-6}{(2x+3y)^2} \end{array}\right\}$$

b) $f(x;y) = \operatorname{arctg}\frac{y}{x}$

$$\left\{\begin{array}{c} \frac{\partial f}{\partial x} = -\frac{y}{x^2+y^2}; \frac{\partial f}{\partial y} = \frac{x}{x^2+y^2}; \\ \frac{\partial^2 f}{\partial x^2} = \frac{2xy}{(x^2+y^2)^2}; \frac{\partial^2 f}{\partial y^2} = \frac{-2xy}{(x^2+y^2)^2}; \frac{\partial^2 f}{\partial x \partial y} = \frac{y^2-x^2}{(x^2+y^2)^2} \end{array}\right\}$$

4. Vypočtěte parciální derivace $\frac{\partial f}{\partial x}$ a $\frac{\partial f}{\partial y}$ funkce f a jejich hodnoty v bodě M

a) $f(x;y) = (x-y)e^{\sqrt{x^3+y^2}} - e^{-2}; M[1;0]$ $\left\{\frac{5}{2}e; -e\right\}$

b) $f(x;y) = 2\mathrm{arctg}(-4x^2y + 4xy^2) - \mathrm{arctg}(8); M[1;1]$ $\{-8; 8\}$

c) $f(x;y) = -3\mathrm{arctg}(8x^2y + 2xy^2) + \mathrm{arctg}(4); M[-1;4]$ $\{96; 24\}$

d) $f(x;y) = -3\ln(-2xy + 2x^2y^2) + \ln 7; M[-1;-4]$ $\left\{7; \frac{7}{4}\right\}$

e) $f(x;y) = 7\ln(-9xy + 5x^2y^2) + \ln 3; M[-1;-6]$ $\left\{-17; -\frac{17}{6}\right\}$

f) $f(x;y) = 3\ln(2xy + 5x^2y^2) + \ln 4; M[1;-1]$ $\{8; -8\}$

g) $f(x;y) = -8x^2y \cdot \ln(-8x - 9y) + \ln 3; M[1;-1]$ $\{-64; -72\}$

h) $f(x;y) = 6xy^2 \cdot \ln(-5x - 4y) + \ln 5; M[-1;1]$ $\{30; 24\}$

i) $f(x;y) = 8xy^2 \cdot \ln(-7x - 8y) + \sqrt[3]{3}; M[1;-1]$ $\{-56; -64\}$

j) $f(x;y) = \sqrt{\mathrm{arctg}(y^2 - 2x)} + \mathrm{arctg}(-1); M[0;1]$ $\left\{-\frac{1}{\sqrt{\pi}}; \frac{1}{\sqrt{\pi}}\right\}$

k) $f(x;y) = \ln(x^2 + 3y) + \mathrm{y}^2 \cdot \sin x; M\left[0; \frac{1}{2}\right]$ $\left\{\frac{1}{4}; 2\right\}$

l) $f(x;y) = e^{x^2 \cdot \sin 2y} + \frac{1}{e}; M\left[1; \frac{\pi}{2}\right]$ $\{0; -2\}$

m) $f(x;y) = e^{\sqrt{xy} - x^3y} + e^{\sqrt{2}}; M[1;1]$ $\left\{-\frac{5}{2}; -\frac{1}{2}\right\}$

n) $f(x;y) = (y-x) \cdot \mathrm{arctg}\sqrt{x-y} + \mathrm{arctg}(e^{-2}); M[1;0]$ $\left\{-\frac{\pi+1}{4}; \frac{\pi+1}{4}\right\}$

5. Nalezněte rovnice tečné roviny v dotykovém bodě T k dané ploše.

a) $z = 2x^2 + y^2; T[1;1;?]$ $\{4x + 2y - z - 3 = 0\}$

b) $z = -2x^2 + 3y^2 + x; T[2;-1;?]$ $\{7x + 6y + z - 5 = 0\}$

c) $z = 3(x-1)^2 + 2(y+3)^2 + 7; T[2;-2;?]$ $\{6x + 4y - z + 8 = 0\}$

d) $z = \sqrt{xy}; T[1;1;?]$ $\{x + y - 2z = 0\}$

e) $z = xe^{xy}; T[2;0;?]$ $\{x + 4y - z = 0\}$

f) $z = x\sin(x+y); T[-1;1;?]$ $\{x + y + z = 0\}$

g) $z = \ln(x - 2y); T[3;1;?]$ $\{x - 2y - z - 1 = 0\}$

6. Vypočtěte derivace $\frac{\partial z}{\partial t}$ resp. $\frac{\partial w}{\partial t}$ složených funkcí

a) $z = x^2 + y^2 + xy; \quad x = \sin t; \quad y = e^t$ $\{\sin 2t + e^t(\cos t + \sin t) + 2e^{2t}\}$

b) $z = \cos(x + 4y); \quad x = 5t^4; \quad y = \frac{1}{t}$ $\left\{\left(\frac{4}{t^2} - 20t^3\right)\sin\left(5t^4 + \frac{4}{t}\right)\right\}$

c) $z = \sqrt{1 + x^2 + y^2}; \quad x = \ln t; \quad y = \cos t$ $\left\{\frac{\ln t - t\sin t\cos t}{t\sqrt{1 + \ln^2 t + \cos^2 t}}\right\}$

d) $z = \frac{1}{\mathrm{tg}\frac{y}{x}}$; $x = e^t$; $y = 1 - e^{-t}$ $\left\{\frac{e^t - 2}{e^{2t}\sin^2(e^{-t} - e^{-2t})}\right\}$

e) $w = xe^{\frac{y}{z}}$; $x = \mathrm{t}^2$; $y = 1 - t$; $z = 1 + 2t$ $\left\{e^{\frac{1-t}{1+2t}}\left(2t - \frac{t^2}{1+2t} - \frac{2t^2(1-t)}{(1+2t)^2}\right)\right\}$

f) $w = \ln\sqrt{x^2 + y^2 + z^2}$; $x = \sin t$; $y = \cos t$; $z = \mathrm{tg}\, t$ $\left\{\frac{\sin t}{\cos^3 t(1 + \mathrm{tg}^2 t)}\right\}$

7. Vypočtěte derivace $\frac{\partial z}{\partial s}$ resp. $\frac{\partial z}{\partial t}$ složených funkcí

a) $z = x^2y^3$; $x = \mathrm{s} \cdot \cos t$; $y = \mathrm{s} \cdot \sin t$

$$\{5s^4\cos^2 t \sin^3 t;\ s^5 \sin^2 t \cos t\,(3\cos^2 t - 2\sin^2 t)\}$$

b) $z = \arcsin(x - y)$; $x = s^2 + t^2$; $y = 1 - st$

$$\left\{\frac{2s + t}{\sqrt{1 - (s^2 + t^2 - 1 + st)^2}};\frac{2t + s}{\sqrt{1 - (s^2 + t^2 - 1 + st)^2}}\right\}$$

c) $z = \sin\theta \cdot \cos\varphi$; $\theta = \mathrm{s}\,t^2$; $\varphi = s^2t$

$$\{t^2\cos st^2 \cos s^2t - 2st \sin st^2 \sin s^2t;\ 2st \cos st^2 \cos s^2t - s^2 \sin st^2 \sin s^2t\}$$

d) $z = e^{x+2y}$; $x = \frac{s}{t}$; $y = \frac{t}{s}$

$$\left\{e^{x+2y}\left(\frac{1}{t} - \frac{2t}{s^2}\right); e^{x+2y}\left(\frac{2}{s} - \frac{s}{t^2}\right)\right\}$$

e) $w = e^r\cos\theta$; $r = st$; $\theta = \sqrt{s^2 + t^2}$

$$\left\{e^{st}\left(t\cos\sqrt{s^2 + t^2} - \frac{s\sin\sqrt{s^2 + t^2}}{\sqrt{s^2 + t^2}}\right); e^{st}\left(s\cos\sqrt{s^2 + t^2} - \frac{t\sin\sqrt{s^2 + t^2}}{\sqrt{s^2 + t^2}}\right)\right\}$$

f) $w = \mathrm{tg}\frac{u}{v}$; $u = 2s + 3t$; $v = 3s - 2t$

$$\left\{\frac{2 - \frac{6s + 9t}{3s - 2t}}{(3s - 2t)\cos^2\frac{2s + 3t}{3s - 2t}};\frac{3 - \frac{4s + 6t}{3s - 2t}}{(3s - 2t)\cos^2\frac{2s + 3t}{3s - 2t}}\right\}$$

8. Nalezněte derivaci funkce $f(x;y)$ v daném bodě B a ve směru určeném úhlem φ.

a) $f = x^3y^4 + x^4y^3; B[1;1];\ \varphi = \frac{\pi}{6}$ $\{9{,}56\}$

b) $f = ye^{-x}; B[0;4];\ \varphi = \frac{2\pi}{3}$ $\{2{,}87\}$

c) $f = e^x \cos y\,; B[0;0];\ \varphi = \frac{\pi}{4}$ $\{0{,}71\}$

9. Nalezněte gradient funkce $f(x;y)$, gradient funkce $f(x;y)$ v bodě B a derivaci funkce $f(x;y)$ v daném bodě B a ve směru určeném jednotkovým vektorem $\vec{u}$.

a) $f = \sin(2x+3y)\,; B[-6;4];\ \vec{u} = \left(\frac{\sqrt{3}}{2}; -\frac{1}{2}\right)$ $\{0{,}23\}$

b) $f = \frac{y^2}{x}; B[1;2];\ \vec{u} = \left(\frac{2}{3}; \frac{\sqrt{5}}{3}\right)$ $\{0{,}31\}$

10. Nalezněte gradient funkce $f(x;y)$, gradient funkce $f(x;y)$ v bodě B a derivaci funkce $f(x;y)$ v daném bodě B a ve směru určeném vektorem $\vec{u}$.

a) $f = e^x \sin y\,; B\left[0; \frac{\pi}{3}\right];\ \vec{u} = (-6;8)$ $\{-0{,}12\}$

b) $f = \frac{x}{x^2+y^2}; B[1;2];\ \vec{u} = (3;5)$ $\{-0{,}08\}$

c) $f = x^4 - x^2y^3; B[2;1];\ \vec{u} = (1;3)$ $\{-2{,}53\}$

d) $f = \frac{1}{\mathrm{tg}(xy)}; B[1;2];\ \vec{u} = (5;10)$ $\{-2{,}16\}$

11. Určete a klasifikujte stacionární body daných funkcí $f(x;y)$

a) $f(x;y) = x^2 - 2xy + 2y$ $\{[1;1;1]SB\}$

b) $f(x;y) = x^2 + xy + y^2 + y$ $\left\{\left[\frac{1}{3}; -\frac{2}{3}; -\frac{1}{3}\right] LMin\right\}$

c) $f(x;y) = xy - 2x - 2y - x^2 - y^2$ $\{[-2;-2;4]LMax\}$

d) $f(x;y) = y^3 + 3x^2y - 6x^2 - 6y^2 + 2$

$\{[0;0;2]OLMax; [0;4;-30]OLMin; [2;2;-14]SB; [-2;2;-14]SB\}$

e) $f(x;y) = xy(1-x-y)$

$\{[1/2;0;0], [0;1/2;0]\mathit{nelze\ rozhodnout}; [1;0;0]SB; [0;1;0]SB\}$

f) $f(x;y) = x^3 - 12xy + 8y^3$

$\{[0;0;0]SB; [2;1;-8]LMin\}$

g) $f(x;y) = xy + \frac{1}{x} + \frac{1}{y}$ $\{[1;1;3]LMin\}$

h) $f(x;y) = e^y(y^2 - x^2)$ $\left\{[0;0;0]SB; \left[0;-2;\frac{4}{e^2}\right] LMax\right\}$

12. Vypočtěte integrály, v případě potřeby zaokrouhlujte na 3 DM:

a) $\int_0^3 \int_1^2 x^2 y dy dx$ a $\int_1^2 \int_0^3 x^2 y dx dy$ $\left\{\frac{27}{2}\right\}$

b) $\int_1^4 \int_0^2 (6x^2 y - 2x) dy dx$ $\{222\}$

c) $\int_0^1 \int_1^2 (4x^3 y - 9x^2 y^2) dy dx$ $\left\{-\frac{11}{2}\right\}$

d) $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \int_{-1}^5 \cos y \, dy dx$ $\{-0{,}123\}$

e) $\int_{-3}^3 \int_0^{\frac{\pi}{2}} (y + y^2 \cos x) dx dy$ $\{18\}$

f) $\int_1^3 \int_1^5 \frac{\ln y}{xy} dy dx$ $\{1{,}423\}$

g) $\int_0^2 \int_0^4 (y^3 e^{2x}) dy dx$ $\{1715{,}141\}$

h) $\int_1^4 \int_1^2 \left(\frac{x}{y} + \frac{y}{x}\right) dy dx$ $\{7{,}278\}$

i) $\int_0^1 \int_0^3 e^{x+3y} dx dy$ $\{121{,}419\}$

13. Vypočtěte integrály:

a) $\int_0^4 \int_0^{\sqrt{y}} xy^2 dx dy$ $\{32\}$

b) $\int_0^1 \int_{2x}^2 (x - y) dy dx$ $\{-1\}$

c) $\int_0^1 \int_{x^2}^x (1 + 2y) dy dx$ $\left\{\frac{3}{10}\right\}$

d) $\int_0^2 \int_y^{2y} \mathrm{x}\, y \, dx dy$ $\{6\}$

e) $\int_0^1 \int_0^{s^2} \cos s^3 \, dt ds$ $\left\{\frac{1}{3}\sin 1 \doteq 0{,}28\right\}$

f) $\int_0^1 \int_0^{e^v} \sqrt{1 + e^v} dw dv$ $\{2{,}894\}$

14. Vypočtěte integrály:

a) $\iint_D y^2 dA, D = \{[x; y] \in R^2;\ -1 \leq y \leq 1\ \wedge\ -y - 2 \leq x \leq y\}$ $\left\{\frac{4}{3}\right\}$

b) $\iint_D \frac{y}{x^5+1} dA, D = \{[x; y] \in R^2;\ 0 \leq x \leq 1\ \wedge\ 0 \leq y \leq x^2\}$ $\{0{,}1 \ln 2 \doteq 0{,}069\}$

c) $\iint_D x dA, D = \{[x; y] \in R^2;\ 0 \leq x \leq \pi\ \wedge\ 0 \leq y \leq \sin x\}$ $\{\pi \doteq 3{,}142\}$

d) $\iint_D x^3 dA, D = \{[x; y] \in R^2;\ 1 \leq x \leq e\ \wedge\ 0 \leq y \leq \ln x\}$

$\left\{\frac{1}{16}(3e^4 + 1) \doteq 10{,}3)\right\}$

15. Vypočtěte integrály, kde D je ohraničeno danými křivkami:

a) $\iint_D y dA, D: x = y^2; y = x - 2$ $\left\{\frac{9}{4}\right\}$

b) $\iint_D x \cos y\, dA, D: y = 0; y = x^2; x = 1$ $\{0{,}5(1 - \cos 1) \doteq 0{,}23\}$

c) $\iint_D (x^2 + 2y) dA, D: y = x; y = x^3; x \geq 0$ $\left\{\frac{23}{84} \doteq 0{,}27\right\}$