# APPLIED MATHEMATICS – THE NUMERICAL SOLVING OF THE SYSTEMS OF THE LINEAR EQUATIONS

Lecture Notes

**Calculation of the eigenvalues of the matrices**

We find λ, what is the solution of the matrix equation

Ax = λx

A… squared matrix, order of N

λ… eigenvalue of the matrix A

x… eigenvector of the matrix A

spectrum of the matrix A .. set of all eigenvalues

E… unit matrix

$Ax - \lambda x = 0$

$(A - \lambda E)x = 0$

the homogenous system of the equations – determinant $|A - \lambda E| = 0$

calculations of this determinant we get the polynomial p (λ) of order n – characteristic polynomial

characteristic equation p(λ) = 0 – roots λ … eigenvalues of the matrix A

EX. $A = \begin{pmatrix} 2 & -3 & 1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{pmatrix}$

characteristic equation: $\begin{vmatrix} 2-\lambda & -3 & 1 \\ 1 & -2-\lambda & 1 \\ 1 & -3 & 2-\lambda \end{vmatrix}$

Saruss rule: $(2-\lambda)(-2-\lambda)(2-\lambda) - 3 - 3 — 2 - \lambda + 3(2-\lambda) + 3(2-\lambda) = 0$

$(-4 - 2\lambda + 2\lambda + \lambda^2)(2-\lambda) - 6 + 2 + \lambda + 6(2-\lambda) = 0$

$-8 + 4\lambda + 2\lambda^2 - \lambda^3 - 6 + 2 + \lambda + 12 - 6\lambda = 0$

$-\lambda^3 + 2\lambda^2 - \lambda = 0$

$\lambda^3 - 2\lambda^2 + \lambda = 0$

$\lambda(\lambda^2 - 2\lambda + 1) = 0$

$\lambda(\lambda - 1)^2 = 0 => \ \lambda_1 = 0 \ \lambda_{23} = 1 \ spectrum \ \{0; 1\}$

eigenvector corresponding with eigenvalue $\lambda_1$=0

EVROPSKÁ UNIE
Evropské strukturální a investiční fondy
Operační program Výzkum, vývoj a vzdělávání
MINISTERSTVO ŠKOLSTVÍ, MLÁDEŽE A TĚLOVÝCHOVY

$$\begin{pmatrix} 2-0 & -3 & 1 \\ 1 & -2-0 & 1 \\ 1 & -3 & 2-0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} =$$

$$> \left(\begin{array}{ccc|c} 2 & -3 & 1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & -3 & 2 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 2 & -3 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 3 & -3 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 2 & -3 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

We choose: $x_3 = t$

$x_2 - t = 0 \rightarrow \ x_2 = t$

$2x_1 - 3t + t = 0 \rightarrow \ x_1 = t$

vector: $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} t \\ t \\ t \end{pmatrix} < t\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, eigenvector for $\lambda_1 = 0$

eigenvector corresponding with eigenvalue $\lambda_{23} = 0$

$$\left(\begin{array}{ccc|c} 2-1 & -3 & 1 & 0 \\ 1 & -2-1 & 1 & 0 \\ 1 & -3 & 2-1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -3 & 1 & 0 \\ 1 & -3 & 1 & 0 \\ 1 & -3 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -3 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

We choose: $x_3 = s$

$x_2 = t$

$x_1 = 3t - s$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3t - s \\ t \\ s \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} t + \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} s$$

We choose: t = s = 1

$\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$ eigenvector for $\lambda_{23} = 1$

**THE NUMERICAL SOLUTION OF THE SYSTEM OF THE LINEAR EQUATIONS**

- the most important topic → discretization→linearization

$A_x = b \qquad A = \begin{pmatrix} a_{11} \dots a_{1N} \\ a_{M1} \dots a_{MN} \end{pmatrix} \quad b = \begin{pmatrix} b_1 \\ \vdots \\ b_M \end{pmatrix}$ x= $\begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix}$

solvability → Frobenius teorem $h(A) = n$ Regular matrix ∃! solution

$h(A) < n$ ∞ solutions, parametric

$h(A) \neq h(A|b)$ 0 solutions

→ 2 main groups of the solutions:

**Direct methods** → transformation of the matrices od the system to the Δ matrix

(for fully matrices) → LU decomposition, GEM, GEM – JORDAN MODIFICATION

**Iteration methods** →limit of the series of the vectors

(for thin matrices) → Jacobi methods, Gauss- Seidel, the biggest fallout, gradient, relaxation …

→ Cramer rule → $x = \frac{Dx_1}{D}; \dots ; \frac{Dx_N}{D}$ → for big systems, very difficult for the calculation

→ bad conditioned

**DIRECT METHODS**

**LU decomposition METHOD**

The principle: the decomposition of the matrix A to the product of two Δ matrices $A = L \cdot u$

The Matrix L – all elements of the squared matrix which lays above the main diagonal = 0

U - below main diagonal = 0

$Ax = b$ $A = L \cdot u$

$Lux = b$ firstly we solve: $Ly = b$ y… helping vector

$ux = y$ after it $u \cdot x = y$

- using - when we solve more systems with the same matrix A

**→ Sufficient conditions for LU decomposion**

→ for every squared matrix A, all main sub determinants different from the 0

∃ matrix L and U, such $A = L \cdot u$

→ LU decomposition is given unequivocally
→ 1. diag. elements of the matrix L are equal to 1 (Doolittle decomposion)

→ 2 . diag.elements of the matrix U are equal to 1 (Crout decomposion)

EY:
$$\begin{aligned} 2x_1 + x_2 &= 3 \\ 6x_1 + 6x_2 - 2x_3 &= 5 \\ 4x_1 + 14x_2 - 7x_3 &= -5 \end{aligned}$$

Matrix $\begin{pmatrix} 2 & 1 & 0 \\ 6 & 6 & -2 \\ 4 & 14 & -7 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \\ -3 \end{pmatrix}$

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 6 & 6 & -2 \\ 4 & 14 & -7 \end{pmatrix} = \begin{pmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{pmatrix} \cdot \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix}$$

a) DOOLITTLE $l_{11} = l_{22} = l_{33} = 1$

b) CROUT $u_{11} + u_{22} + u_{33} = 1$

we applied Doolittle

$$\begin{pmatrix} 2 & 1 & 0 \\ 6 & 6 & -2 \\ 4 & 14 & -7 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 4 & 1 \end{pmatrix} \cdot \begin{pmatrix} 2 & 1 & 0 \\ 0 & 3 & -2 \\ 0 & 0 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 4 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 0 \\ 0 & 3 & -2 \\ 0 & 0 & 1 \end{pmatrix}$$

$2 = u_{11}$ $\quad 6 = 2l_{21} \Rightarrow l_{21} = 3$ $\quad 4 = 2l_{31} \Rightarrow l_{31} = 2$

$1 = u_{12}$ $\quad 6 = 3 + u_{12} \Rightarrow u_{12} = 3$ $\quad 14 = 2 + 3l_{32} \Rightarrow l_{32} = 4$

$0 = u_{13}$ $\quad -2 = 0 + u_{23} \Rightarrow u_{23} = -2$ $\quad -7 = 0 - 8 + u_{33} \Rightarrow u_{33} = 1$

$L \cdot y = b$

$$\begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 4 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \\ -5 \end{pmatrix} \qquad \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 3 \\ -4 \\ 5 \end{pmatrix}$$

$y_1 = 3$

$3 \cdot 3 + y_2 = 5 y_2 = -4$ direct substitution

$2 \cdot 3 + 4 \cdot (-4) + y_3 = -5 \quad y_3 = 5$

$u \cdot x = y$

$$\begin{pmatrix} 2 & 1 & 0 \\ 0 & 3 & -2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ -4 \\ 5 \end{pmatrix}$$

$x_3 = 5$

$3x_2 - 2 \cdot 5 = -4 \qquad x_2 = 2$

EVROPSKÁ UNIE
Evropské strukturální a investiční fondy
Operační program Výzkum, vývoj a vzdělávání

MINISTERSTVO ŠKOLSTVÍ, MLÁDEŽE A TĚLOVÝCHOVY

$2x_1 + 2 + 0 = 3 \qquad x_1 = \frac{1}{2}$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 2 \\ 5 \end{pmatrix}$$

**ITERATION METHODS (IM)**

THE NORM OF THE VETOR: $\|\vec{a}\|_1 = \sqrt{a_1^2 + a_2^2 + a_3^2}$ Euclid norm

$\vec{a} = (a_1; a_2; a_3)$ $\|\vec{a}\|_2 = max(|a_1|; \dots ; |a_2|)$

$\|\vec{a}\|_3 = |a_1| + \dots + |a_3|$

THE NORMS OF THE MATRICES

A $\|A\|_1 = \sqrt{\lambda_0}$ $x_0 = max(\lambda_1; \dots ; \lambda_n)$ Spectral NORM

$\|A\|_2 = max_j \, (\sum(a_{ij})$ Column norm

$\|A\|_3 = max_i \, (\sum(a_{ij})$ Row norm

Spectral radius of the matrix $S(A) = \max(|\lambda_1|; |\lambda_2|; \dots ; |\lambda_n|)$

IM→ $Ax = b \; \rightarrow we \; use \; the \; arbitrary \; vector \; x^{(0)} \; \rightarrow x$

$\rightarrow x = Hx + k =; \; x^{(k)} = H. x^{(k-1)} + k$

**JACOBI METHOD** A regular, diagonal dominant $(a_{ij}) > \sum_{i \neq 1}(a_{ij})$

$A = D + L + U$

$D \; diagonal$

$L \; low \; matrix$

$U \; upper \; matrix$

$(D + L + U). x = b$

$Dx = -(L + U)x + b$

$x = -D^{-1}(L + U). x + D^{-1}. b$

EX

Method gradual approximations

$10x_1 + x_2 = 11 \Rightarrow x_1 = 1{,}1 - 0{,}1x_2$

$x_1 + 10 \quad = -9 \Rightarrow x_2 = -0{,}9 - 0{,}1x_1$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & -0{,}1 \\ -0{,}1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1{,}1 \\ -0{,}3 \end{pmatrix}$$

**GAUSS-SAIDEL METHOD** (METHOD OF THE GRADUAL REPAIRS)

A: Symmetric, positive definite

$$(D + L + u)x = b$$

$$(D + L)x + ux = b$$

$$(D + L).\, x = b - u\, .\, x \qquad /.\,(D + L)^{-1}$$

$$x = (D + L)^{-1} - b - (D + L)^{-1}.\, u\, .\, x$$

<u>EX:</u>

$10x_1 + x_2 = 11 \Rightarrow x_1 = 1{,}1 - 0{,}1x_2$ $\qquad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & -0{,}1 \\ -0{,}1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1{,}1 \\ -0{,}3 \end{pmatrix}$

$x_1 + 10 \quad = -9 \Rightarrow x_2 = -0{,}9 - 0{,}1x_1$

$for\ calculation\ we\ use\ x_1\ zcome\ from\ this\ approximation\ (not\ from\ last)$

$$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}^{k+1} = \begin{pmatrix} 0 & -0{,}1 \\ -0{,}1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}^{k+1} + \begin{pmatrix} 0 & -0{,}1 \\ -0{,}1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}^{k} + \begin{pmatrix} 1{,}1 \\ -0{,}3 \end{pmatrix}$$

because $x_1\ we\ know\ from\ 2k + 1\ iteration$

We cannot say whim method has the fasters convergent