APPLIED MATHEMATICS – THE NUMERICAL SOLVING OF THE DIFFERENTIAL EQUATIONS

Lecture Notes

$y' = f(x;y)\,;\ x \in < a;b >$

$=> y = f(x);\ \ y = y(x_0)$ => initial condition

→ Lipschitz condition (∃ and unique of the solution)

If exists such value L, where $\forall\ x\ \in < a;b > and\ u,v \in R\ applies$:

$|f(x;u) - f(x;v)| \leq L|u - v|$, after it ∃ unique solutions of the equation $y = y(x_0)$

→generalization: $\vec{y}' = f(\vec{y};x)$ $\vec{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_i \end{pmatrix};\ y_i = y_i(x)$

→the principle of the numerical methods

The net method (nodes x_0 (=a)…..x_n (=b))

-in every node of the net we find the approximation x

-y_0 – is the first node of the net (we know the x_0)

$y_{i+1} = y_i + \Delta y_i = y_i + h; S(x_i; y_i; h)$

-mistake Δy is groowing a) local - in actual step

b) global – cumulation from the previous steps

=>the net method give the most exact solution in surroundings of the initial node in <a;b>

1. the Euler method

EY → the approximation of the integral curve

$[x_0;y_0]; [x_1;y_1]; \ldots; [x_n;y_n]$

→ the guideline of the abscissa = tangent line k_1 invert curve in the point $[x_i;y_i]$

$k_i = S\,(x_i; y_i; h) = y'(x_i) = f(x_i; y_i)$

$=> y_{i+1} = y_i + h \cdot y'(x_i) + \frac{h^2}{2} \cdot y''(\xi)\ the\ estimation\ of\ the\ mistake\ from\ TP\ we\ neglect$

$y_{i+1} = y_i + h \cdot S(x_i; y_i; h) => \ y_1 = y_0 + h \cdot f(x_0; y_0)$

$y_2 = y_1 + h \cdot f(x_1; y_1)$

$$y_n = y_{n-1} + h \cdot f(x_{n-1}; y_{n-1})$$

$y' = f(x;y) \quad y(x_0) = y_0$

2. Modified Euler method

→ the estimation executed using the rectangular method in <x_i ; x_{i+1}>

$$y_{i+1} - y_i = \int_{x_i}^{x_{i+1}} y'(x)dx = h \cdot f(x_i + \frac{h}{2}; y_i \cdot \left(x_i + \frac{h}{2}\right))$$

$$y_{i+1} = y_i + h \cdot f\left[x_i + \frac{h}{2}; y_i + \frac{h}{2} \cdot f(x_i; y_i)\right]$$

$$y_1 = y_0 + h \cdot f\left[x_0 + \frac{h}{2}; y_0 + \frac{h}{2} \cdot f(x_0; y_0)\right]$$

$$y_2 = y_1 + h \cdot f\left[x_1 + \frac{h}{2}; y_1 + \frac{h}{2} \cdot f(x_1; y_1)\right]$$

3. Euler-Cauchy method

→ the estimation executed using the trapezoidal method

$$y_{i+1} - y_i = \int_{x_i}^{x_{i+1}} y'(x)dx = \frac{h}{2} \cdot [f(x_i; y_i) + f(x_i + h; y(x_i + h))]$$

$$y(x_i + h) = y(x_i) + h \cdot y'(x_i) = y(x_i) + h \cdot f(x_i; y_i)$$

$$y_{i+1} - y_i = \frac{h}{2} \cdot \left[f(x_i; y_i) + f\big(x_i + h; y_i + h \cdot f(x_i; y_i)\big)\right]$$

$$y_{i+1} = y_i + \frac{h}{2} \cdot \left[f(x_i; y_i) + f\big(x_i + h; y_i + h \cdot f(x_i; y_i)\big)\right]$$

$$y_1 = y_0 + \frac{h}{2} \cdot \left[f(x_0; y_0) + f\big(x_0 + h; y_0 + h \cdot f(x_0; y_0)\big)\right]$$

$$y_2 = y_1 + \frac{h}{2} \cdot \left[f(x_1; y_1) + f\big(x_1 + h; y_1 + h \cdot f(x_1; y_1)\big)\right]$$

4. the Runge-Kutta method (Matlab)

$y_{i+1} = y_i + h \cdot S(x_i; y_i)$; $S(x_i; y_i) = \gamma_1 \cdot k_1 + \gamma_2 \cdot k_2 + \cdots + \gamma_r \cdot k_r$

$$k_1 = f(x_i; y_i)$$

$$k_2 = f(x_i + \alpha_2 \cdot h; y_i + \beta_2 \cdot k_1)$$

S… direction of the directional function for the one-step methods higher orders → we gain as the combination of the different guidelines k_1, k_2 …k_r

γ_i … ordinal coefficient (from MNO)

α,β…coefficients, they are given with the regard to the size r

R-K 4th order: $k_1 = f(x_i; y_i)$

$$k_2 = f\left[x_i + \frac{h}{2}; y_i + h \cdot \frac{k_1}{2}\right]$$

$$k_3 = f\left[x_i + \frac{h}{2}; y_i + h \cdot \frac{k_2}{2}\right]$$

$$k_4 = f\left[x_i + \frac{h}{2}; y_i + h \cdot k_3\right]$$

$$S = \frac{1}{6} \cdot (k_1 + 2 \cdot k_2 + 2 \cdot k_3 + k_4)$$

$$y_{i+1} = y_i + h \cdot S$$

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Numerical solutions of the systems ODE1.O

Matrix notation: $\begin{pmatrix} y'_1 \\ y'_2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ f(x) \end{pmatrix}$

Euler method:

$$\overrightarrow{y_{(i+1)}} = \overrightarrow{y_{(i)}} + h \cdot \vec{f}(x_i; y_i)$$

We rewrite: $y_{1(i+1)} = y_{1(i)} + h \cdot f_1(x_i; y_{1(i)}; y_{2(i)})$

Only 2 equations: $y_{2(i+1)} = y_{2(i)} + h \cdot f_2(x_i; y_{1(i)}; y_{2(i)})$

Transfer LODE2.O to the system ODE1.O

EX: a)

$y'' + 2 \cdot y' - y = 0$

$(y = y_1)' \rightarrow y' = y_1' = y_2$

$(y = y_2)' \rightarrow y'' = y_2'$

DOZ: $y_2' + 2 \cdot y_2 - y_1 = 0$

$y_1' = y_2$

$y_2' = -y_1 - 2 \cdot y_2$

Matrix: $\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

b) $y'' + 2 \cdot y' - y = \mathrm{f(x)}$

DOZ: $y_2' + 2 \cdot y_2 - y_1 = \mathrm{f(x)}$

$y_1' = y_2$

$y_2' = -y_1 - 2 \cdot y_2 + \mathrm{f(x)}$

Matrix: $\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ f(x) \end{pmatrix}$