APPLIED MATHEMATICS – THE NUMERICAL SOLUTION OF THE NONLINEAR EQUATIONS

Lecture Notes

→ to find all zero points of the function: f(x)=0

Nonzero point: f ($\overline{x}$)=0 →:$\overline{x}$:

→ the formulas exists only into 4th degree

→ using of the formulas is problematic too

1. Step: Separation of the roots – finding as short as possible intervals,(there does not exists algorithm) in which lies just one root (generally complex)
2. Step: Approximation of the roots – finding of the numerical solving of the equation
   a) number, which represents the approximate root value with the given accurancy …. $\overline{x}$.
   b) Algorithm, which can find $\overline{x}$.

<a;b> from separation

→ Numerical computing $\overline{x}$ is based in construction of the series $\{x_n\}_{n=0}^{\infty}$ such that $\lim_{n\to\infty} x_n = \overline{x}$.

→ Numerical solving – such member $\{x_n\}_{n=0}^{\infty}$, which satisfies $|x_k - \overline{x}| < \varepsilon$

→efficiency of the methods is given by speed of the convergence (elected accuracy).

→Teoretical base provides the Banach theorem about the fixed point (sufficient condition for ∃ the fixed point of the function ω=> ω($\overline{x}$)= $\overline{x}$) fixed point = root.

Let ω:<a;b>-><a;b>. Let function ω(x) is contractive in <a;b>.

(∃ value K ϵ <0;1) such, that the pair of the values $x_1 \neq x_2;\ x_1, x_2 \epsilon < a; b >$ holds:

$(\omega(x_1) - \omega(x_2)) \leq k|x_1 - x_2|$). Then ∃! Fixed point $\overline{x}$ function ω(x) in <a;b>.

$\omega(\overline{x}) = \overline{x}$ $\quad |\omega_{(a)} - \omega_{(b)}| < k|a - b|$ k ϵ (0;1)

- Contractility of the function $\varphi(x)$ means, that the distance of the images is less then the distance of the patterns – is necessary for ∃ of fixed point
- criterion for stopping of the calcukation

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$|\bar{x} - x_k| \leq \frac{k^k}{1-k}$ enough is $|\bar{x} - x_k| < \varepsilon$

Simple iteration method (SIM)

f(x) = 0 |+ x we choose $x_0 \in < a; b >$

f(x) + x = x $\varphi(x) = x$ and suppose $x_{k+1} = \varphi(x_k)$

Construction of $\varphi(x)$ is not unique (we have more possibilities) →always we have to control of the contractility

$< a; b > \exists\, q < 1;\ |f'(x)| < q\ \forall\, \mathrm{x} \in < \mathrm{a}; \mathrm{b} > \rightarrow$ function is contractive

EX.

$x^3 + x - 1000 = 0$ int. $< 9{,}10 > \rightarrow < 0; 271 >$

Construction of the $\varphi(x)$:

1. $x = 1000 - x^3$

$x = \varphi(x)$ $\varphi_1(x) = 1000 - x^3$ $\varphi'_1(x) = -3x^2$ $\varphi'_1(10) = -300 \nless 1$ Inpossible

2. $x = \sqrt[3]{1000 - x}$

$\varphi_2(x) = \sqrt[3]{1000 - x}$ $\varphi'_1(x) = -\frac{1}{3}\frac{1}{\sqrt[3]{(1000-x)^2}}$ sure is < 1 pro $\forall\, x \in < 9; 10 >$

$< 9; 10 > \rightarrow < 9{,}9\overline{6}; 9{,}9699 > \subset < 9; 10 > \cup$

Computation

$x_0 = 10$

$x_1 = \varphi_2(x_0) = 9{,}966554934$

$x_2 = \varphi_2(x_1) = 9{,}966667166$

$x_3 = \varphi_2(x_2) = 9{,}966666789$

$x_4 = \varphi_2(x_3) = 9{,}966666791$

$x_5 = \varphi_2(x_4) = 9{,}966666791$ – the same as previous $\varphi_2(x_4) = x_5 = x_4$ fixed point of the mapping

Contraction coefficient (generally method) $\boldsymbol{x \in < a; b >}$

$f(x) = 0 \rightarrow \varphi(x) = x$

We suppose $f(x)$ is differentiable in $< a; b >$ and holds:

$0 < C \leq f'_{(x)} \leq d$ in all $< a; b >$

$f(x) = 0$

$x + f(x) = x \rightarrow x = x - \lambda f(x)$ λ ... we find such function $\varphi(A) = x - \lambda f(x)$ is contractive

$\varphi(x) = x - \lambda f(x)$

$\varphi'(x) = x - \lambda f'(x) \rightarrow |\varphi'(x)| \leq k < 1$

$1 - k \leq \lambda \cdot c \leq \lambda f'(x) \leq \lambda \cdot d \leq 1 + k \rightarrow \lambda = \dfrac{2}{c + d}$

Without AV

$1 - q \leq \lambda \cdot c$

$\underline{1 + q \leq \lambda \cdot d}$

$2 = \lambda(c + d) \rightarrow \lambda = \dfrac{2}{c + d}$

Iteration formula: $\boldsymbol{x_{k+1} = x_k - \frac{2}{c+d} \cdot f(x_k)}$

EX: in $< 1; 2 >$ solve the eq. $x^3 + 4x^2 - 10 = 0$ with accuracy k = 10^{-4}. How many iterations we have to find (rounded to GDM) $10^{-4} = 1 \cdot 10^{-4} = 0{,}0001$

$x^3 + 4x^2 - 10 = 0$

We rewrite in shape: $\varphi(x) = x - \lambda f(x)\ and\ \lambda$ ...find such function $\varphi(x)$ is constrictive

$f(x) = x^3 + 4x^2 - 10$

$f'(x) = 3x^2 + 8x$ $\rightarrow 3x \cdot \left(x + \frac{8}{3}\right)$ in interval $< 1; 2 >$ is fce grooving

$f'(1) = 11$ $f'(2) = 28 \rightarrow$ Holds values between 8 and 28

$\rightarrow \lambda = \frac{2}{11+28} = \frac{2}{39} \rightarrow \varphi(x) = x - \frac{2}{39}(x^3 + 4x^2 - 10)$

We choose $x_0 = 1{,}5$

$x_0 = 1{,}5$	→ A	$\lvert x_{n+1} - x_n \rvert$ $\lvert B - A \rvert$	
$x_1 = \varphi(x_0) = 1{,}378205$	→ B → B → A	0,121795	X
$x_2 = \varphi(x_1) = 1{,}367147$	→B → B → A	0,011058	X
$x_3 = \varphi(x_2) = 1{,}365521$	→ B → B → A	0,001625	X
$x_4 = \varphi(x_3) = 1{,}365275$	→ B → B → A	0,000247	X
$x_5 = \varphi(x_4) = 1{,}365237$	→ B	0,000038	Ano

The approximate value of the root is $x_5 = \tilde{x} = 1{,}36524$ with accuracy 0,0001

The Newton method

- the most known and used: function have to be a concave or convex

$(b) = f(b) \cdot (x - b)$

x-axis: $[x_1; 0]$

$f(b) = f(b) \cdot (x_1 - b) = x_1 - b \rightarrow \ x_1 = \frac{f_{(b)}}{f'_{(b)}}$

$y - f(x_1) = f'(x_1) \cdot (x - x_1)$

x-axis: $[x_2; 0]$

$-f(x_1) = f'(x_1) \cdot (x_2 - x_1) \rightarrow \ x_2 = x_1 - \frac{f_{(x_1)}}{f'_{(x_1)}}$

$\ldots \rightarrow \{x_0; x_1; x_2; \ldots\} \rightarrow \alpha$

Sufficient condition of convergence:

$< a; b >$ $x_0 = a$; je-li $f'(x) \cdot f''(x) < 0$

$x_0 = b$; je-li $f'(x) \cdot f''(x) > 0$

<u>EX:</u> Calculation of $\sqrt{2}$ $f(x) = 0$ $< a; b > = < 1; 2 >$

$\varepsilon = 0{,}0001$ $x^2 - 2 = 0$

$f(x) = x^2 - 2$

$f'(x) = 2x$

$f''(x) = 2$ in interval $< 1; 2 >$ applies $f'^{(x)} * f''^{(x)} > 0 \rightarrow x_0 = 2$

$$x_{k+1} = x_k - \frac{x_k^2 - 2}{2x_k} = A - \frac{A^2 - 2}{2A}$$

$x_0 = 2$	→ A	$\lvert B - A \rvert$	
$x_1 = 1{,}5$	→ B → B → A	0,5	X
$x_2 = 1{,}41667$	→B → B → A	0,08333	X
$x_3 = 1{,}414216$	→ B → B → A	0,002451	X
$x_4 = 1{,}414214$	→ B	0,000002	yes

$\sqrt{2} = 1{,}141421$ with accuracy 0,0001

The secant method

- NMT – the fastest method
- disadvantage – we have to calculate the derivation (it could be very difficult)

- in iteration formula from NMT we substitute f'(x_k) by the numerical derivation

$f'(x_k) \approx \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}}$

Iteration formula $x_{k+1} = x_k - f(x_k)\frac{x_k - x_{k-1}}{f(x_k) - f(x_{k-1})}$

disadvantage: to calculate x_{k+1} we have to use x_k and x_{k-1} (two-steps-method) (NMT is one-step)

EX. $x - cosx = 0$ V $< 0; \frac{\pi}{2} >$ $\varepsilon = 0{,}001$

$$x_{k-1} = A - C\frac{A - B}{C - D}$$

firstly save memory A->B ! and after it E – A

0,7391 is the root with accuracy 0,001

Next methods: halving the interval, regula falsi etc.