Rozvoj kvality vzdělávání, hodnocení a strategického řízení na Univerzitě Pardubice
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APPLIED MATHEMATICS – THE NUMERICAL INTEGRATION

Lecture Notes

Numerical integration

We do not know the explicit sharp of the function

We know it but it is too complex

Approximate calculation of the integral

Methods are based to replacement of the function using the interpolation polynomial

The general formulation of the problem

$$\int_a^b f(x)\,dx\,;a<b\in\mathbb{R}$$

Newton-Cotes quadrature formulas of the closed type – nodes – end points

Newton-Cotes quadrature formulas of the open type – nodes – centers

Newton-Cotes quadrature formulas of the closed type NCKV

$\langle a;b\rangle$ is divided into equidistant nodes

$$a=x_0<x_1<\cdots<x_n=b$$

$$hr=\frac{b-a}{n}$$

In every interval $\langle x_{i-1};x_i\rangle, i=1;\dots;n$ we replace the integrand by the Lagrange interpolation polynomial (LIP) order of *k*

$$L_{i,k}\colon\int_{x_{i-1}}^{x_i} f(x)\,dx\approx\int_{x_{i-1}}^{x_i} L_{i,k}(x)\,dx$$

Simple NCKV order *k*

In interval $\langle a;b\rangle$ applies the composite NCKV

$$\int_a^b f(x)\,dx\approx\sum_{i=1}^{n}\int_{x_{i-1}}^{x_i} f(x)\,dx\approx\sum_{i=1}^{n}\int_{x_{i-1}}^{x_i} L_{i,k}(x)dx$$

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Determination of the mistake $R_k(f)$ na $\langle x_{i-1}; x_i \rangle$

$$R_{1,k}(f) = \int_{x_{i-1}}^{x_i} f(x)\,dx - \int_{x_{i-1}}^{x_i} L_{i,k}(x)dx = \int_{x_{i-1}}^{x_i} \frac{f^{(k+1)}(\eta_i)}{(k+1)!}(x-t_0)\dots(x-t_k)dx$$

t_k inside nodes for construction LIP

$f^{(k+1)}(\eta_i)$ we estimate $M = \max_{x\in\langle a;b\rangle} \left|f^{(k)}(x)\right|$

Rectangular method (k = 0) NCKV

$f(x)$ we replace in every sub interval by the polynomial of the zero order (constant)

in $\langle x_{i-1}; x_i \rangle$ $\quad f(x) \curvearrowright L_{i,0} = f(x_{i-1})$

$$\int_{x_{i-1}}^{x_i} f(x)dx \approx \int_{x_{i-1}}^{x_i} f(x_{i-1})dx = f(x_{i-1}) \cdot [x]_{x_{i-1}}^{x_i} = f(x_{i-1})$$

in $\langle a; b \rangle$

$$\int_a^b f(x)dx \approx \sum_{i=1}^n \int_{x_{i-1}}^{x_i} L_{i,0}(x)dx = \sum_{i=1}^n \int_{x_{i-1}}^{x_i} f(x_{i-1})dx = h\sum_{i=1}^n f(x_{i-1}) = h(f_0 + f_1 + \dots + f_{n-1})$$

composite rectangular rule

mistake

$$R_0(f) = \int_a^b f(x)dx - h\sum_{i=1}^n f(x_{i-1}) = \sum_{i=1}^n \int_{x_{i-1}}^{x_i} \frac{f'(\eta)}{1!}(x-x_{i-1})dx = \sum_{i=1}^n \frac{f'(\eta)}{1!}\int_{x_{i-1}}^{x_i}(x-x_{i-1})dx$$

$$= \left| \begin{matrix} \text{substitution } h = \dfrac{b-a}{n} \\ x = x_{i-1} + th;\ t \in \langle 0; 1 \rangle \\ dx = h\,dt \end{matrix} \right| = \sum_{i=1}^n M_i \frac{h^2}{2}$$

$$\int_{x_{i-1}}^{x_i} (x-x_{i-1})dx = \int_0^1 th \cdot h\,dt = h^2 \int_0^1 t\,dt = h^2 \left[\frac{t^2}{2}\right]_0^1 = \frac{1}{2}h^2$$

$$x - x_{i-1} = th$$

$$dx = h\,dt$$

$$\text{for } x = x_{i-1}: 0 = th \rightarrow t = 0$$

$$\text{for } x = x_i: \underbrace{x_i - x_{i-1}}_{h} = th$$

$$h = th$$

$$h - th = 0$$

$$h(1-t) = 0 \rightarrow t = 1$$

in $\langle a; b\rangle$ $|R_0(f)| \le M_1 \cdot \frac{h^2}{2} \cdot n = M_1 \cdot \frac{(b-a)^2}{2n}$

$$M_1 = \max_{x\in\langle a;b\rangle} |f'(x)|$$

EX: {exact value: $[-\cos x]_0^{\frac{\pi}{2}} = -\cos\frac{\pi}{2} + \cos 0 = 0 + 1 = \underline{\underline{1}}$

$$\int_0^{\frac{\pi}{2}} \sin x\, dx$$

x	$\sin x$
0	0
$\frac{\pi}{8}$	0,3827
$\frac{\pi}{4}$	0,7071
$\frac{3\pi}{8}$	0,9239
$\frac{\pi}{2}$	1

$$\int_0^{\frac{\pi}{2}} \sin x\, dx \doteq \frac{\pi}{8}(0 + 0{,}3827 + 0{,}7071 + 0{,}9239) \doteq 0{,}7884$$

The estimation of the mistake:

$$M_1 \cdot \frac{\left(\frac{\pi}{2} - 0\right)^2}{2 \cdot 4} = 1\,\frac{\left(\frac{\pi}{2}\right)^2}{8} = \frac{\pi^2}{4} \cdot \frac{1}{8} = \frac{\pi^2}{32} \doteq 0{,}3084$$

$$f(x) = \sin x$$

$$f'(x) = \cos x \rightarrow \max \text{ na } \langle 0; \frac{\pi}{2} \rangle \text{ je } 1$$

[=> exact value lies in (0,7884 ± 0,3084)]

Trapezoidal method (NCKV of the 1st order)

We replace in every $\langle x_{i-1}; x_i \rangle, i = 1; \dots; n\ f(x)$

$$L_{i,1}(x) = f(x_{i-1})\frac{x - x_i}{x_{i-1} - x_i} + f(x_i)\frac{x - x_{i-1}}{x_i - x_{i-1}}; i = 1; \dots; n$$

in $x \in \langle x_{i-1}; x_i \rangle$

$$L_{i,1}(x); h = x_i - x_{i-1}$$

$$\left.\begin{matrix}[x_{i-1}; f(x_{i-1})] \\ [x_i; f(x_i)]\end{matrix}\right\} L_{i,1}(x) = f(x_{i-1}) \cdot \frac{x - x_i}{x_{i-1} - x_i} + f(x_i)\frac{x - x_{i-1}}{x_i - x_{i-1}}$$

$$\int_{x_{i-1}}^{x_i} f(x)dx \doteq \int_{x_{i-1}}^{x_i} \left[f(x_{i-1})\frac{x - x_i}{x_{i-1} - x_i} + f(x_i)\frac{x - x_{i-1}}{x_i - x_{i-1}}\right] dx$$

$$= \frac{f(x_{i-1})}{\underbrace{x_{i-1} - x_i}_{-h}} \int_{x_{i-1}}^{x_i} (x - x_i)dx + \frac{f(x_i)}{\underbrace{x_i - x_{i-1}}_{h}} \int_{x_{i-1}}^{x_i} (x - x_{i-1})dx$$

$$= \left| \begin{matrix} \text{substitution: } x = x_{i-1} + th; t \in \langle 0; 1 \rangle \\ dx = h\, dt \\ \text{pro } x = x_{i-1}: x_{i-1} = x_{i-1} + th \Rightarrow t = 0 \\ \text{pro } x = x_i: x_i = x_{i-1} + th \\ x_i - x_{i-1} = th \\ h = th \\ h(1 - t) = 0 \Rightarrow t = 0 \end{matrix} \right|$$

$$= \left| \begin{matrix} \text{for mistake: } \oplus \\ x - x_{i-1} = x_{i-1} + th - x_{i-1} \\ x - x_i = x_{i-1} + th - x_i = th - h = h(t - 1) \end{matrix} \right|$$

$$= -\frac{f(x_{i-1})}{h}\int_0^1 (x_{i-1}+th-x_i)h\,dt + \frac{f(x_i)}{h}\int_0^1 (x_{i-1}+th-x_{i-1})h\,dt$$

$$= -\frac{f(x_{i-1})}{h}\int_0^1 \big(h(t-1)\big)h\,dt + \frac{f(x_i)}{h}\int_0^1 (th)h\,dt$$

$$= -\frac{f(x_{i-1})}{h}\cdot h^2\int_0^1 (t-1)dt + \frac{f(x_i)}{h}h^2\int_0^1 t\,dt$$

$$= -hf(x_{i-1})\left[\frac{t^2}{2}-t\right]_0^1 + hf(x_i)\left[\frac{t^2}{2}\right]_0^1$$

$$= -hf(x_{i-1})\left[\left(\frac{1}{2}-1\right)-(0-0)\right] + hf(x_i)\left[\frac{1}{2}-\frac{0}{2}\right] = \frac{h}{2}f(x_{i-1}) + \frac{h}{2}f(x_i)$$

$$= \frac{h}{2}[f(x_{i-1})+f(x_i)]$$

in $\langle a; b\rangle$

$$\int_a^b f(x)dx \approx \sum_{i=1}^{n}\frac{h}{2}[f(x_{i-1})+f(x_i)]$$

$$= \frac{h}{2}\{{}_{i=1}[f(x_0)+f(x_1)] + {}_{i=2}[f(x_1)+f(x_2)] + \cdots + {}_{i=n-1}[f(x_{n-2})+f(x_{n-1})]$$

$$+ {}_{i=n}[f(x_{n-1})+f(x_n)]\} = \frac{h}{2}\left[f(x_0) + 2\sum_{i=2}^{n-1} f(x_i) + f(x_n)\right]$$

$$= \frac{h}{2}\left[f(a) + 2\sum_{i=2}^{n-1} f(x_i) + f(b)\right]$$

The estimation of the mistake:

$$R_{i,1}(f) = \frac{f''(\eta)}{2!}\int_{x_{i-1}}^{x_i}(x-x_{i-1})(x-x_i)dx = |\text{the same substitution}| = \frac{f''(\eta)}{2}\int_0^1 th\cdot h(t-1)h\,dt \quad \oplus$$

$$= \frac{f''(\eta)}{2}\cdot h^3\int_0^1 (t^2-t)dt = \frac{f''(\eta)}{2}h^3\left[\frac{t^3}{3}-\frac{t^2}{2}\right]_0^1 = \frac{f''(\eta)}{2}h^3\left[\frac{1}{3}-\frac{1}{2}\right]$$

$$= -\frac{1}{12}h^3 f''(\eta) = -\frac{1}{12}h^3 M_2$$

$$|R_1(f)| \le n\cdot\frac{1}{12}h^3 M_2 = \frac{n\cdot M_2}{12}\frac{(b-a)^3}{n^3} = \frac{M_2}{12n^2}(b-a)^3;\ M_2 = \max_{x\in\langle a;b\rangle}|f''(x)|$$

EX:

$$\int_0^{\frac{\pi}{2}} \sin x \, dx$$

x	$\sin x$
0	0
$\frac{\pi}{8}$	$0{,}3827$
$\frac{\pi}{4}$	$0{,}7071$
$\frac{3\pi}{8}$	$0{,}9239$
$\frac{\pi}{2}$	1

$$\int_0^{\frac{\pi}{2}} \sin x \, dx \doteq \frac{\frac{\pi}{8}}{2}(0 + 2(0{,}3827 + 0{,}7071 + 0{,}9239) + 1) \doteq 0{,}9871$$

The estimation of the mistake:

$$\frac{M_2}{12n^2}(b-a)^3 = \frac{M_2}{12 \cdot 4^2} \cdot \left(\frac{\pi}{2} - 0\right)^3 = \frac{1}{12 \cdot 16} \cdot \frac{\pi^3}{8} = \frac{\pi^3}{1536} \doteq 0{,}0202$$

$$f(x) = \sin x$$

$$f'(x) = |\cos x|$$

$$f''(x) = |-\sin x| \Rightarrow M_2 = 1$$

[=> exact value lies in (0,9871 ± 0,0202)]

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Summary NCKV *k*-th order

k	Name	Simple	Composed	Mistake
0	Rectangle method	$h \cdot f(x_{i-1})$	$h\sum_{i=1}^{n} f(x_{i-1})$	$M_1 \cdot \frac{b-a}{2n}$ $M_1 = \max_{x\in\langle a;b\rangle} \lvert f'(x)\rvert$
1	Trapezoidal method	$\frac{h}{2}\left(f(x_{i-1}) + f(x_i)\right)$	$\frac{h}{2}\left[f(a) + 2\sum_{i=2}^{n-1} f(x_i) + f(b)\right]$	$\frac{M_2}{12n^2}(b-a)^3$ $M_2 = \max_{x\in\langle a;b\rangle} \lvert f''(x)\rvert$
2	Simpson method	$\frac{h}{3}\left(f(x_{i-1}) + 4f(s) + f(x_i)\right)$ s střed $x_{i-1} - x_i$	$\frac{h}{3}\left[f(x_0) + 2\sum_{i=1}^{n-1} f(x_{2i}) + 4\sum_{i=1}^{n} f(x_{2i-1}) + f(x_{2n})\right]$	$\frac{M_4}{2880n^4}(b-a)^5$ $M_4 = \max_{x\in\langle a;b\rangle} \lvert f^{IV}(x)\rvert$

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