APPLIED MATHEMATICS – FUNDAMENTALS OF THE FINANCIAL MATHEMATICS

Lecture Notes

Sequence

Geometrical sequence (GS)

Sequence $\{a_n\}^{\infty}{}_{n=1}$ is called GEOMETRIC if there exists real number q, that complies $a_{n+1} = a_n \cdot q$.

The number q is called the quotient of the GC

$\{a_1;\ a_2;\ a_3;\ a_4; \cdots; a_{n,}\}$ $\qquad$ $a_n = a_1 q^{n-1}$

$a_1\ \ a_1$q $\ a_1 q^2\ a_1 q^{n-1}$ $\qquad$ $a_r = a_s q^{r-s}$

➔ Sum s_n (of the first n components) GS $\{a_n\}_{n=1}^{\infty}$ with quotient q_n

a) $q = 1 \rightarrow\ s_n = na_1$ $\qquad$ Proof: $s_n = a_1 + a_1 + \cdots a_1 = na_1$

b) $q \neq 1 \rightarrow\ s_n = a_1 \frac{q^n - 1}{q-1}$ $\qquad$ Proof: $s_n = a_1 + a_1 q + a_1 q^2 + \cdots + a_1 q^{n-1}$ $/\cdot q$ I.

$$q s_n = a_1 q + a_1 q^2 + a_1 q^3 + \cdots + a_1 q^n \qquad \text{II.}$$

II. – I. $s_n q - s_n = -a_1 + a_1 q - a_1 q + a_1 q^2 - a_1 q^2 + \cdots + a_1 q^{n-1} - a_1 q^{n-1} + a_1 q^n$

$$s_n(q-1) = a_1 q^n - a_1 = a_1(q^n - 1)$$

$$s_n = a_1 \frac{q^n - 1}{q-1}$$

➔ The comparison of the AS x GS

AP	GP
$a_n = a_{n-1} + d$ $a_n = a_1 + (n-1)d$ $a_s = a_r + (s-r)d$ $s_n = \frac{n}{2}(a_1 + a_n)$	$a_n = a_{n-1}\, q$ $a_n = a_1 q^{n-1}$ $a_s = a_r\ q^{s-r}$ $s_n = a_1 \frac{q^n - 1}{q - 1}, (q \neq 1)$

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Utilization of the GS

A) Growth (decline)

Initial value a_0 beginning of the 1st period a_0

growth p (%) end of the 1st period $a_1 = a_0\left(1+\frac{p}{100}\right)$

expected value a_n after n periods end 2nd period $a_2 = a_1\left(1+\frac{p}{100}\right) = a_0\left(1+\frac{p}{100}\right)^2$

end n. period $a_n = a_0\left(1+\frac{p}{100}\right)^n$

$\left\{a_0;\ a_0\ \left(1+\frac{p}{100}\right);\ a_0\ \left(1+\frac{p}{100}\right)^2;\cdots;\ a_n\left(1+\frac{p}{100}\right)^n\right\}$ GP, first itemn a_0 and quotient $q = 1+\frac{p}{100}$

B) Compound interest

Interest measure 1st per. $p\%$

Interest period

principal J_0

tax 15% (generally $d\%$)

initial J_0

after 1st per. interest $J_0\frac{p}{100}$

tax: $J_0\frac{p}{100}\frac{d}{100}$

value: $J_0 + J_0\frac{p}{100} - J_0\frac{p}{100}\frac{d}{100} = J_0\left(1+\frac{p}{100}-\frac{p}{100}*\frac{d}{100}\right)$

$$J_1 = J_0\left[1+\frac{p}{100}\left(1-\frac{d}{100}\right)\right]$$

in 2nd period

$J_1\frac{p}{100}$ interest

tax $J_0\frac{p}{100}\frac{d}{100}$

$$J_2 = J_1 + \frac{p}{100} - J_1\frac{p}{100}\frac{d}{100} = J_1\left(1+\frac{p}{100}-\frac{p}{100}\frac{d}{100}\right) = J_1\left[1+\frac{p}{100}\left(1-\frac{d}{100}\right)\right]$$

$$J_2 = J_0\left[1+\frac{p}{100}\left(1-\frac{d}{100}\right)\right]\left[1+\frac{p}{100}\left(1-\frac{d}{100}\right)\right] = J_0\left[1+\frac{p}{100}\left(1-\frac{d}{100}\right)\right]^2$$

After n periods

$$J_n = J_0 \left[1 + \frac{p}{100}\left(1 - \frac{d}{100}\right)\right]^n$$

C) Saving

The same value I_0 every interest period (in the end as well!)

Interest p %
tax d %(15 %)

after n periods S

1st value is interested n times
2nd value is interested $(n-1)$ times
3rd. value is interested $(n-2)$ times

$$S = I_0\left[1 + \left(1 - \frac{d}{100}\right) * \frac{p}{100}\right]^n + I_0\left[1 + \left(1 - \frac{d}{100}\right) * \frac{p}{100}\right]^{n-1} + \cdots + I_0\left[1 + \left(1 - \frac{d}{100}\right) * \frac{p}{100}\right] + I_0$$

↓ a_{n+1} ↓ a_n ↓ a_2 ↓ a_1

$$S = I_0 \frac{\left[1 + \left(1 - \frac{d}{100}\right)\frac{p}{100}\right]^{n+1} - 1}{\left[1 + \left(1 - \frac{d}{100}\right)\frac{p}{100}\right] - 1} = I_0 \frac{\left[1 + \left(1 - \frac{d}{100}\right)\frac{p}{100}\right]^{n+1} - 1}{\left(1 - \frac{d}{100}\right)\frac{p}{100}}$$

D) The repayment of the loan

Ex: At the biginning of 1995 a bank garanted a loan of 1 000 000 Kč for 3 years with an annual interest of 14 % (interest period of 1 year). Installment once a year, first after one year. How much is the installment?

installment S
debt in the end of 1995 (+ interest) $10^6(1-0{,}14)$
installment S
debt in the beginning of 1996 (after 1st installment) $10^6(1-0{,}14) - S$

debt in the beginning of 1997 (+ interest – 2nd. installment)

$$[10^6(1-0{,}14) - S](1+0{,}14) - S = 10^6\,(1+0{,}14)^2 - S(1+0{,}14) - S$$

debt in the beginning of 1998 (+interest – 3rd. installment)

$$[10^6\,(1+0{,}14)^2 - S(1+0{,}14) - S](1+0{,}14) - S =$$

$$= 10^6(1+0{,}14)^3 - S\,(1+0{,}14)^2 - S\,(1+0{,}14) - S$$

The loan will be repayed in the end of 1998 → debt = 0

$$\rightarrow \quad 10^6(1+0{,}14)^3 - S\,(1+0{,}14)^2 - S\,(1+0{,}4) - S = 0$$

GS we replace by the sum

$$10^6(1+0{,}14)^3 - S\,\frac{(1+0{,}14)^3-1}{(1+0{,}14)-1} = 0$$

$S = \frac{10^6(1+0{,}14)^3\cdot 0{,}14}{(1+0{,}14)^3-1} \cong 430731{,}48\,Kč$

Generalization: loan D, n interested periods, interest of p percent, n equal installment

$$S = \frac{D\left(1+\frac{P}{100}\right)^n \frac{p}{100}}{\left(1+\frac{p}{100}\right)^n - 1}$$

E) **Continuous interest**

Ex: amount $1000\ Kč$ is every year interested by 10% (end of the year 1100 Kč)

- halfyear inscribing interest 1000 $(t=0)$ 1050 $(t=1/2)$ 1102,2 $(t=1)$
- quarteryear inscribing interest 1000 1025 1050,625 1076,89 1103,81

$1000\ (t=0)\quad 1025\ (t=1/4)\quad 1050{,}625\ \left(t=\frac{1}{2}\right), 89\ (t=3/4)\quad 1103{,}81\ (t=1)$

- etc. -> continuous interest

$y(t)$ … amount in Kč in time t $\qquad y(Kč)\quad t(years)$

- year interest t≥1 $\quad y(t+1) = y(t) + \frac{y(t)}{10}; t \geq 1$

  $y(t)$ in time $t\epsilon\langle 0;1\rangle$ is given by the initial value

  ->when $t\epsilon\langle 0;1\rangle\ \ y(t) = 1000$

  -> $y(1) = y(0) + \frac{y(0)}{10} = 1000 + \frac{1000}{10} = 1100$

  t=0

- $\frac{1}{2}$ year interest $\ t \geq 1/2\quad y(t+1/2) = y(t) + \frac{y(t)}{20};\ t \geq 1/2$

  ->when $t\epsilon\langle 0;1/2\rangle\ \ y(t) = 1000$

$$y(1/2) = y(0) + \frac{y(0)}{20} = 1000 + \frac{1000}{20} = 1050$$

$$y(1) = y(1/2) + \frac{y(1/2)}{20} = 1050 + \frac{1050}{20} = 1102{,}2$$

- Etc.:
- For continuous interest we define h…length of the period (in years) forassign interest (early was $h=1$; $h=1/2$)

$t \geq h \;\; y(t+h) = y(t) + \frac{y(t)}{10} h \rightarrow \frac{y(t+h)-y(t)}{h} = \frac{y(t)}{10};\; y(t) = 1000\, p for\; t\epsilon\langle 0; h)$

- Continuous interest $h \rightarrow 0$ limit changeovr:

$$\lim_{h\to 0} \frac{y(t+\mathrm{h})-\mathrm{y(t)}}{h} = \lim_{h\to 0} \frac{\mathrm{y(t)}}{10} \rightarrow \frac{dy}{dt}(t) = \frac{\mathrm{y(t)}}{10};\; t \geq 0;\; y(0) = 1000$$

- DE. $y' = \frac{y}{10};\; y(0) = 1000$ initial problem
- Generalization: value I_0 interest p % $y(0) = I_0$

$$y' = \frac{1}{p}\, y$$

$$\frac{d(y)}{dt} = \frac{y}{p}$$

$$\int \frac{dy}{y} = \int \frac{dt}{y}$$

$$\ln|y| = \frac{1}{p}\, t + c$$

OS: $y = e^{\frac{1}{p}t+C} = C \cdot e^{\frac{1}{p}t}$ IC. $y(0) = 1000$ $I_0 = C \cdot e^{\frac{1}{p}0} \rightarrow C = I_0$

PS: $y = I_0 \cdot e^{\frac{1}{p}t}$

For our ex. $y(1) = 1000\, e^{\frac{1}{10}*1} = 1105{,}17$ kč

If there is the tax d % we have to count with the constant $\frac{1-d}{100}$

$$y = I_0\, e^{\frac{100-d}{100}\cdot\frac{1}{p}\cdot t}$$

F) The natural growth (decline) rule of the population

We suppose: ideal conditions (infinite space, food, without predators, immunity against to the infections)

t time (independent variable)

P trhe number of the members in the population (dependent variable)

The rate of the growth of the population … time derivation $\frac{dP}{dt}$ is linearly related to the population (times by the constant)

=> $\frac{dP}{dt} = kP$ $\;k$ is constant

$\int \frac{dP}{P} = = \int k \cdot dt$

$\ln|P| = kt + C$

$P = Ae^{kt}; P(0) = P_0; A = P_0 \rightarrow P = P_0 e^{kt}$

impact 1 EQ is possible to rewrite $\frac{1}{p} \cdot \frac{dP}{dt} = k$ relative rate of the growth is constant

impact 2 Emigration: the rate of the emigration … m

the rate of the changeing of the population is $\frac{dP}{dt} = kP - m$

G) The logistic model

Real models have to react to the limited sources and space

-start; exponential growth -> critical capacity M

2 assumptions -> $\frac{dP}{dt} \approx kP$ (when P is small -> the rate of the growth is related to P)

-> $\frac{dP}{dt} < 0$ when P>M

$\frac{dP}{dt} = kP(1 - \frac{P}{M})$

$\int \frac{dP}{P(1-\frac{P}{M})} = \int \frac{M}{P(\mathrm{M-P})}\,\mathrm{dP} = \int \frac{1}{P}\,\mathrm{dP} + \int \frac{1}{\mathrm{M-P}}\,\mathrm{dP} = \ln|P| - \ln|M - P|$

$\frac{M}{P(\mathrm{M-P})} = \frac{a}{\mathrm{P}} + \frac{b}{M-P} = \frac{aM - aP + \mathrm{bP}}{P(M-P)} \rightarrow M{:}\, a = 1; P{:}\, 0 = -(a + b) = -1 + b \rightarrow b = 1$

$$\int \frac{dP}{P(1 - \frac{P}{M})} = \int k\,dt$$

$\ln|P| - \ln|M - P| = kt + C$

$$\ln\left|\frac{P}{M - P}\right| = kt + C$$

$\frac{P}{M-P} = e^{kt+C} = Ae^{kt}$

$P + PAe^{kt} = AMe^{kt}$

$P(1 + \mathrm{A}\, e^{kt}) = AMe^{kt}$

$$P = \frac{\mathrm{AM}e^{kt}}{1 + \mathrm{A}e^{kt}} = \frac{\mathrm{M}}{\frac{1}{\mathrm{A}\, e^{kt}} + 1} = \frac{\mathrm{M}}{1 + \frac{1}{A}e^{-kt}} = \frac{\mathrm{M}}{1 + Be^{-kt}}$$

B const. $t = 0; P = P_0$

$P_0 = \frac{\mathrm{M}}{1+Be^0}$ $\quad P_0 + P_0 B = M \rightarrow B = \frac{\mathrm{M - P_0}}{\mathrm{P_0}}$

Solution og the LM:

$$P(t) = \frac{\mathrm{M}}{1 + Be^{-kt}}\,;\; B = \frac{\mathrm{M} - \mathrm{P}_0}{\mathrm{P}_0}$$

$$\lim_{t\to\infty} \frac{\mathrm{M}}{1 + \frac{\mathrm{M} - \mathrm{P}_0}{\mathrm{P}_0} e^{-kt}} = M$$

H) The other growth models

➔ $\frac{dP}{dt} = kP\left(1 - \frac{\mathrm{P}}{M}\right) - C$ C the regular substraction

➔ $\frac{dP}{dt} = kP(1 - \frac{\mathrm{P}}{M})\,(1 - \frac{\mathrm{m}}{P})$ m minimal value of the dying of the species

I) Sigmoidal curve of the growth

$\frac{dN}{dt} = rN\left(1 - \frac{N}{k}\right)$ $\quad 0 \le t \le 10 \to sigmoidal\ curve\ of\ the\ growth$

$r = 1; k = 1\ N(0) = 0{,}01\ (FISH\ per\ m^3)$

$\frac{N}{dt} = rN\left(\frac{k-N}{K}\right)$ $\quad \int \frac{dN}{N(1-N)} = \int \frac{1}{N} dN + \int \frac{1}{1-N} dN = ln|N| - ln|1-N| + C$

$\frac{dN}{dt} = \frac{r}{k} N(K - N)$ $\quad \frac{1}{N(1-N)} = \frac{A}{N} + \frac{B}{1-N} = \frac{A1-AN+BN}{N(1-N)} = \frac{N(B-A)+A1}{N(1-N)}$

$\int \frac{dN}{N(k-N)} = \int \frac{r}{k} dt$ $\quad B - A = 0 \quad B = A - 1 \quad A = 1$

$\int \frac{dk}{N(1-N)} = \int dt$ $\quad Ex: 0{,}01 = \frac{1}{1+De^{0}{}_{1}} = \frac{1}{1+D}$

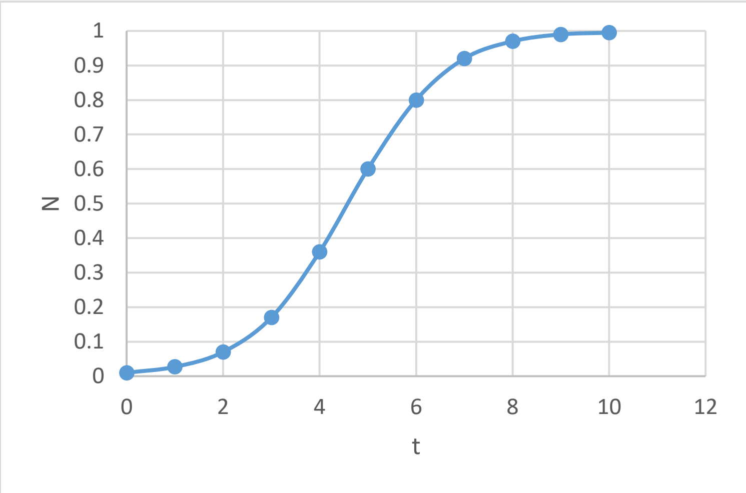
$ln\left|\frac{N}{1-N}\right| = Ce^t$ $\quad 1 + D = 100 \quad D = 99$

$N = Ce^t - CNe^t$ $\quad PS: N = \frac{1}{1+99e^{-t}}$

$N(1 + Ce^t) = Ce^t$

$$N = \frac{Ce^t}{1 + Ce^t} = \frac{1}{\frac{1}{C}e^{-t} + 1} = \frac{1}{1 + De^{-t}}$$

$$GS: N = \frac{1}{1 + 0e^{-t}}$$



t	0	1	2	3	4	5	6	7	8	9	10
N	0,01	0,027	0,07	0,17	0,36	0,6	0,8	0,92	0,97	0,99	0,995