APPLIED MATHEMATICS – THE APPROXIMATION OF THE FUNCTIONS

Lecture Notes

The approximation of the functions

- Replacement of the complex functions using suitable function (easier)
- Types of the Approximation

Interpolation (function comes through all poles)

$$\varphi(x_i) = f_i; i = 1, 2, \dots, n$$

Uniform approximation (is not unique)

$$max|f(x) - \varphi(x)|;\ x \in \langle x_0;\ x_n \rangle$$

The last squares method

Interpolation

General formulation of the task

N+1 values are given

x_0	f_0
x_1	f_1
⋮	
x_{N-1}	f_{N-1}
x_N	f_N

$$[x_i;\ f_i]; i = 0 \dots N$$

Poles

We find φ(x) → in the dependency of the shape of the function φ(x) we speak about interpolation:

- Polynomial
- Spline
- Trigonometric
- Exponencial

The classes of the approximate functions $\varphi(x; a_0; a_1; \dots; a_n)$

x … variable

a_i; i = 0 … n coefficients

Algebraic polynomials

$$P_n(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n$$

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- We can easy I and D

Weierstrasse theorem about the approximation

If f(x) is continuous function which is given in $x \in < a; b >$ and $\varepsilon > 0$, then ∃ polynomial P(x) given in <a; b> such that

$$|f(x) - P(x)| < \varepsilon;\ \forall x \in < a; b >$$

If the points [x_i; y_i] are given; i= 0 … n such, that $x_i \neq x_j$ then ∃ unique polynomial $P_n(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ at most n-th degree with property $P_n(x_i) < y_i; i = 1 \dots n$

$$P_n(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$$

$$[x_0; y_0] \in P_n(x): y_0 = a_0 + a_1x_0 + a_2x_0^2 + \cdots + a_nx_0^n$$

$$[x_1; y_1] \in P_n(x): y_1 = a_0 + a_1x_1 + a_2x_1^2 + \cdots + a_nx_1^n$$

$$[x_2; y_2] \in P_n(x): y_2 = a_0 + a_1x_2 + a_2x_2^2 + \cdots + a_nx_2^n$$

$$[x_n; y_n] \in P_n(x): y_n = a_0 + a_1x_n + a_2x_n^2 + \cdots + a_nx_n^n$$

n+1 equations of n+1 unknowns (a_0 … a_n)

for $x_0 < x_1 < x_2 < \dots < x_n$

$$D = \begin{vmatrix} 1 & x_0 & x_0^2 & \dots & x_0^{n-1} & x_0^n \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & x_n & x_n^2 & \dots & x_n^{n-1} & x_n^n \end{vmatrix} = (x_2 - x_1) \cdot (x_3 - x_2) \cdot (x_4 - x_3) \cdot \dots \cdot (x_n - x_{n-1}) \neq 0$$

Vandermond D

→ ∃! solution

Cramer rule: $\left(a_0 = \frac{D_0}{D}; a_1 = \frac{D_1}{D}; \dots; a_n = \frac{D_n}{D}\right)$

The indefinite method

- Badly conditioned system
- Big computing difficulty for groowing

Lagrange IP

Indirect construction using the fundamental polynomials $l_i(x); i = 0, \dots n$

$l_i(x_i) = 0$ pro $i \neq j$

$l_i(x_i) = 1$ pro $i = j$

Polynomials $l_i(x)$ have roots

$$x_0, x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n \rightarrow l_i(x) = C_i \cdot (x - x_0)(x - x_1) \dots (x - x_{i-1})(x - x_{i+1}) \dots (x - x_n)$$

for $l_i(x_i) = 1$

$$C_i = \frac{1}{(x_i - x_0)(x_i - x_1)\dots(x_i - x_{i-1})(x_i - x_{i+1})\dots(x_i - x_n)}$$

After substitution we get:

$$l_i(x) = \frac{(x - x_0)(x - x_1)\dots(x - x_{i-1})(x - x_{i+1})\dots(x - x_n)}{(x_i - x_0)(x_i - x_1)\dots(x_i - x_{i-1})(x_i - x_{i+1})\dots(x_i - x_n)}$$

$$L_n(x) = \sum_{i=0}^{n} y_i l_i(x)$$

EX: Construct LIP for function F(x) given using the table

X_i	0	1	2	5
Y_i	2	3	12	147

Fundamental pln: $l_0(x) = \frac{(x-1)(x-2)(x-5)}{(0-1)(0-2)(0-5)} = \dots = -\frac{1}{10}(x^3 - 8x^2 + 17x - 10)$

$$l_1(x) = \frac{(x-0)(x-2)(x-5)}{(1-0)(1-2)(1-5)} = \dots = \frac{1}{4}(x^3 - 7x^2 + 10x)$$

$$l_2(x) = \frac{(x-0)(x-1)(x-5)}{(2-0)(2-1)(2-5)} = \dots = -\frac{1}{6}(x^3 - 6x^2 + 5x)$$

$$l_3(x) = \frac{(x-0)(x-1)(x-2)}{(5-0)(5-1)(5-2)} = \dots = \frac{1}{60}(x^3 - 3x^2 + 2x)$$

$$L_3(x) = 2\left[-\frac{1}{10}(x^3 - 8x^2 + 17x - 10)\right] + 3\left[\frac{1}{4}(x^3 - 7x^2 + 10x)\right] + 12\left[-\frac{1}{6}(x^3 - 6x^2 + 5x)\right] + 147\left[\frac{1}{60}(x^3 - 3x^2 + 2x)\right] = x^3 + x^2 - x + 2$$

Newton interpolation polynomial (NIP)

LIP – theoretical importance (after changing input data (even a single point) ⇒ recounting)

Proportional difference

<table>
<tr><th>X</th><th>F(x)</th><th>$\Delta^1 f(x)$</th><th>$\Delta^2 f(x)$</th><th>…</th><th>$\Delta^{n-1} f(x)$</th><th>$\Delta^n f(x)$</th></tr>
<tr><td>X_0</td><td>F_0</td><td rowspan="2">$f[x_0; x_1]$</td><td rowspan="3">$f[x_0; x_1; x_2]$</td><td rowspan="14">…</td><td rowspan="7">$f[x_0; \ldots; x_{n-1}]$</td><td rowspan="14">$f[x_0; \ldots; x_n]$</td></tr>
<tr><td rowspan="2">X_1</td><td rowspan="2">F_1</td></tr>
<tr><td rowspan="2">$f[x_1; x_2]$</td></tr>
<tr><td rowspan="2">X_2</td><td rowspan="2">F_2</td><td rowspan="2">$f[x_1; x_2; x_3]$</td></tr>
<tr><td rowspan="2">$f[x_2; x_3]$</td></tr>
<tr><td rowspan="2">X_3</td><td rowspan="2">F_3</td><td rowspan="2">$f[x_2; x_3; x_4]$</td></tr>
<tr><td rowspan="2">$f[x_3; x_4]$</td></tr>
<tr><td rowspan="2">X_4</td><td rowspan="2">F_4</td><td rowspan="4">⋮</td><td rowspan="7">$f[x_1; \ldots; x_n]$</td></tr>
<tr><td rowspan="2">⋮</td></tr>
<tr><td rowspan="2">⋮</td><td rowspan="2">⋮</td></tr>
<tr><td rowspan="2">$f[x_{n-2}; x_{n-1}]$</td></tr>
<tr><td rowspan="2">X_{n-1}</td><td rowspan="2">F_{n-1}</td><td rowspan="3">$f[x_{n-2}; x_{n-1}; x_n]$</td></tr>
<tr><td rowspan="2">$f[x_{n-1}; x_n]$</td></tr>
<tr><td>X_n</td><td>F_n</td></tr>
</table>

$\Delta^0 f(x) = f_i$
$\Delta^1 f(x) =$

- $\frac{f_1 - f_0}{x_1 - x_0} = f[x_0; x_1]$
- $\frac{f_2 - f_1}{x_2 - x_1} = f[x_1; x_2]$
- ....
- $\frac{f_n - f_{n-1}}{x_n - x_{n-1}} = f[x_{n-1}; x_n]$

$\Delta^2 f(x) =$

- $f[x_0; x_1; x_2] = \frac{f[x_1;x_2] - f[x_0;x_1]}{x_2 - x_0}$
- $f[x_1; x_2; x_3] = \frac{f[x_2;x_3] - f[x_1;x_2]}{x_3 - x_1}$
- …
- $f[x_{n-2}; x_{n-1}; x_n] = \frac{f[x_{n-1};x_n] - f[x_{n-2};x_{n-1}]}{x_n - x_{n-2}}$

$$\Delta^n f(x) = f[x_0; \dots; x_n] = \frac{f[x_1; \dots; x_n] - f[x_0; \dots; x_{n-1}]}{x_n - x_0}$$

NIP: $N_n(x) = f_0 + f[x_0; x_1](x - x_0) + f[x_0; x_1; x_2](x - x_0)(x - x_1) + \cdots + f[x_0; x_1; \dots; x_n](x - x_0)(x - x_1) \dots (x - x_{n-1})$

The examle of the calculation of the coefficients for equidistant points $x_i = x_0 + i * h, i = 0, \dots, n$ $[x_i; f_i]$

$N_N(x_i) = f_i \rightarrow [x_0; f_0] : N_0(x_0) = a_0 = f_0$

$$[x_1; f_1] : N_1(x_1) = a_0 + a_1(x_1 - x_0) = f_1 \rightarrow a_1 = \frac{f_1 - a_0}{x_1 - x_0} = \frac{f_1 - f_0}{h}$$

$$[x_2; f_2] : N_2(x_2) = a_0 + a_1(x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1) = f_2$$

$$a_2 = \frac{f_2 - a_0 - a_1(x_2 - x_0)}{(x_2 - x_0)(x_2 - x_1)} = \frac{f_2 - f_0 - \frac{f_1 - f_0}{h}(2h)}{2h\,h} = \frac{f_2 - f_0 - 2f_1 + 2f_0}{2h^2} = \frac{f_2 - 2f_1 + f_0}{2h^2}$$

$$[x_3; f_3] : N_3(x_3) = a_0 + a_1(x_3 - x_0) + a_2(x_3 - x_0)(x_3 - x_1) +$$
$$+ a_3(x_3 - x_0)(x_2 - x_0)(x_1 - x_0) = f_3$$

$$a_3 = \frac{f_3 - a_0 - a_1(x_3 - x_0) - a_2(x_3 - x_0)(x_3 - x_1)}{(x_3 - x_0)(x_2 - x_0)(x_1 - x_0)} =$$
$$\frac{f_3 - f_0 - \frac{f_1 - f_0}{h}(3h) - \frac{f_2 - 2f_1 + f_0}{2h^2}(3h)(2h)}{3h\,2h\,h} = \frac{f_3 - f_0 - 3f_1 + 3f_0 - 3f_2 + 6f_1 - 3f_0}{6h^3} = \frac{f_3 - 3f_2 + 3f_1 - f_0}{6h^3}$$

$[x_K; f_K] : a_K = \frac{\Delta_0^K}{K! h^K}$ $K = 0 \dots n$ Proportional difference

LIP = NIP other construction, the same result

Estimation of the mistake of the LIP and NIP

It comes from the Taylor theorem → the expression of the residuum $f_{(x)} \rightarrow L_n$

$$f_{(x)} = L_{n(x)} + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0) \dots (x - x_n)$$

$$f_{(x)} - L_{n(x)} = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0) \dots (x - x_n)$$

$$\left|f^{(n+1)}(x)\right| \leq M \quad for\ x \in \langle x_0; x_n \rangle$$

EX:

x	0	$\frac{\Pi}{6}$	$\frac{\Pi}{4}$	$\frac{\Pi}{3}$
sinx	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$

→ $L_3(x)$ sinx→cosx→-sinx→-cosx→sinx

M estimation max. value of 4th derivation

M=1 $x \in (0; \frac{\Pi}{3})$

For instance: for $x = \frac{\Pi}{5}$ $L_3\left(\frac{\Pi}{5}\right) \approx \sin(\frac{\Pi}{5})$

$$\left|\sin\frac{\Pi}{5} - L_3\left(\frac{\Pi}{5}\right)\right| \leq \frac{\sin(\xi)}{(3+1)!}(\frac{\Pi}{5} - 0)(\frac{\Pi}{5} - \frac{\Pi}{6})(\frac{\Pi}{5} - \frac{\Pi}{4})(\frac{\Pi}{5} - \frac{\Pi}{3})$$

SPLINES

Smoothness of the binding is important

→ polynomial of the 3rd degree

→ for functional values

$$\varphi_0(x_0) = f_0 \ldots \varphi_{i-1}(x_{i-1}) = f_{i-1}$$

$$\varphi_{i-1}(x_i) = f_i = \varphi_i(x_i)\ldots\varphi_i(x_{i+1}) = f_{i+1} = \varphi_{i+1}(x_{i+1})\ldots = \varphi_{n-1}(x_n) = f_n$$

→ for first derivations

$$\varphi_0'(x_0) = f_0' \ldots \varphi'_{i-1}(x_{i-1}) = f'_{i-1}$$

$$\varphi'_{i-1}(x_i) = f'_i = \varphi'_i(x_i)\ldots\varphi'_i(x_{i+1}) = f'_{i+1} = \varphi'_{i+1}(x_{i+1})\ldots = \varphi'_{n-1}(x_n) = f'_n$$

→ for second derivations

$$\varphi_0''(x_0) = f_0'' \ldots \varphi''_{i-1}(x_{i-1}) = f''_{i-1}$$

$$\varphi''_{i-1}(x_i) = f''_i = \varphi''_i(x_i)\ldots\varphi''_i(x_{i+1}) = f''_{i+1} = \varphi''_{i+1}(x_{i+1})\ldots = \varphi''_{n-1}(x_n) = f''_n$$

$\varphi_i(x)$ is polynomial of the 3rd degree $i = 0, 1\ldots,\ n-1$

EX:

	x	$f_{(x)}$
x_0	0	1
x_1	1	0
x_2	2	2
x_3	3	1

$\varphi_i(x) = a_1 + b_1 x + c_1 x^2 + d_1 x^3$ (unknowns a_1 b_1 c_1 d_1)

$\varphi'_i(x) = b_1 + 2c_1x + 3d_1x^2$ (unknowns $b_1\ c_1\ d_1$)

$\varphi''_i(x) = 2c_1 + 6d_1x$ (unknowns $c_1\ d_1$)

$\varphi_i(x_{i-1}) = f_{i-1}$ $\quad$ $\varphi'_i(x_i) = \varphi'_{i+1}(x_i)$ $\quad$ + initial and final condition

$\varphi_i(x_i) = f_i$ $\quad$ $\varphi''_i(x_i) = \varphi''_{i+1}(x_i)$ $\quad$ $\varphi'_1(x_0) = \varphi''_n(x_n)$

Last Squares method

Approximation function does not to leave through poles → we can eliminate mistakes of the measure (→ aproximation)

general formulation of the problem:
φ(x) f given by the functional values (poles)

x	$f(x)$
x_0	f_0
x_1	f_1
$\vdots$	$\vdots$
x_n	f_n

$\varphi(x) = a_0\varphi_0(x) + a_1\varphi_1(x) + \cdots + a_n\varphi_n(x)$

$\varphi_j(x) \in \tau$ ⋯ class of the functions of certain type

$j = 0 \cdots k$

EX. $\{1; x; x^2; \cdots; x^k\}$; $\{1; \sin x; \cos x; \sin 2x; \cdots\}$; $\left\{1; \frac{1}{x}; \frac{1}{x^2}; \cdots\right\} \cdots$

the condition of the least square method

$$\sum_{i=0}^{n}[\varphi(x_i) - f_i]^2 = S(a_0; a_1; \cdots; a_k)$$

We are finding the coeficients $a_0 \cdots a_k$ such the function $S(a_0; \cdots; a_k)$ acquires the minimal value.

Mineralization of S

$$\frac{\partial S}{\partial a_j} = 0;\ \ j = 0 \cdots k$$

$$2 \cdot \sum_{i=0}^{n} [\varphi(x_i) - f_i] \cdot \varphi_j(x_i) = 0 \qquad /:2$$

$$\sum_{i=0}^{n} [a_0\varphi_0(x_i) + \cdots + a_k\varphi_k(x_i) - f_i] \cdot \varphi_j(x_i) = 0$$ System k+1 equations with k+1 unknowns

$$a_0 \sum_{i=0}^{n} \varphi_0(x_i)\varphi_j(x_i) + \cdots + a_k \sum_{i=0}^{n} \varphi_k(x_i)\varphi_j(x_i) = \sum_{i=0}^{n} f_i\varphi_j(x_i)$$

We put (coefficient) (coefficient) (coefficient) as a scalar product

$$(\varphi_p; \varphi_g) \sum_{i=0}^{n} \varphi_p(x_i)\varphi_g(x_i)$$

We get so-called Normal system of equations

$$a_0(\varphi_0;\varphi_0) + a_1(\varphi_1;\varphi_0) + \cdots + a_k(\varphi_k;\varphi_0) = (f_i;\varphi_0)$$

……………………………

$$a_0(\varphi_0;\varphi_k) + a_1(\varphi_1;\varphi_k) + \cdots + a_k(\varphi_k;\varphi_k) = (f_i;\varphi_k)$$

→if the functions $\varphi_0(x)$ … $\varphi_k(x)$ are linear independent at the set of the nodes, the system has unique solution → $\varphi(x) = a_0\varphi_0(x) + a_1\varphi_1(x) + \cdots + a_k\varphi_k(x)$

EX: Find the pln. Of the 1st degree, which approximate the function f(x) given using the table

We are finding polynomial: $P_1(x) = a_0\varphi_0(x) + a_1\varphi_1(x)$

$\varphi_0(x) = 1;\ \varphi_1(x) = x; f(x_{i)} = y_i$

1	0	1	2	3
Xi	2	4	6	8
Yi	2	11	28	40

→normal system of equations:

$$a_0(1;1) + a_1(x;1) = (f;1)$$

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$$a_0(1;x) + a_1(x;x) = (f;x)$$

Scalar products:

$$(1;1) = \sum_{i=0}^{3}(1;1) = 1*1 + 1*1 + 1*1 + 1*1 = 4$$

$$(x;1) = \sum_{i=0}^{3}(x_i;1) = 2*1 + 4*1 + 6*1 + 8*1 = 20\,;(= (1;x) = \sum_{i=0}^{3}(1;x)$$

$$(x;x) = \sum_{i=0}^{3}(x_i;x_i) = 2*2 + 4*4 + 6*6 + 8*8 = 120\,;(= (1;x^2)$$

$$(f;1) = \sum_{i=0}^{3}(f_i;1) = 2*1 + 11*1 + 28*1 + 40*1 = 81$$

$$(f;x) = \sum_{i=0}^{3}(f_i;x_i) = 2*2 + 11*4 + 28*6 + 40*8 = 536$$

$$\rightarrow 4a_0 + 20a_1 = 81 \qquad \rightarrow a_0 = -12{,}5$$

$$20a_0 + 120a_1 = 536 \;\rightarrow\; a_1 = 6{,}55$$

$$P_1(x) = -12{,}5 + 6{,}55x$$

Note: pln. 3rd degree $P_3(x) = a_0 + a_1x + a_2x^2 + a_3x^3$

$$a_0(1;1) + a_1(x;1) + a_2(x^2;1) + a_3(x^3;1) = (f;1)$$
$$a_0(1;x) + a_1(x;x) + a_2(x^2;x) + a_3(x^3;x) = (f;x)$$
$$a_0(1;x^2) + a_1(x;x^2) + a_2(x^2;x^2) + a_3(x^3;x^2) = (f;x^2)$$
$$a_0(1;x^3) + a_1(x;x^3) + a_2(x^2;x^3) + a_3(x^3;x^3) = (f;x^3)$$

If we use the ORTOGONAL SYSTEM of functions (OSF) in set $\{x_0; x_1; \dots; x_n\}$;
$(\varphi_i)_{i=1}^{N}; (\varphi_i;\varphi_j) = 0\; pro\; i \neq j$
=> *diagonal system*

$$a_0(\varphi_0;\varphi_0) + 0 + \cdots + 0 = (f;\varphi_0)$$
$$0 + a_1(\varphi_1;\varphi_1) + \cdots 0 = (f;\varphi_1)$$
……………………
$$0 + 0 + \cdots a_k(\varphi_k;\varphi_k) = (f;\varphi_k)$$

The Ralston construction:

We give: $\varphi_{-1} = 0;\; \varphi_0 = 1;\; b_0 = 0$

$$\varphi_{k+1} = (x - C_R) * \varphi_k - b_k * \varphi_{k-1}$$

$$a_k = \frac{(x*\varphi_k;\varphi_k)}{(\varphi_k;\varphi_k)}\;;\; b_k = \frac{(\varphi_k;\varphi_k)}{(\varphi_{k-1};\varphi_{k-1})}$$

EX. The Ralston construction

i	0	1	2	3
X_i	2	4	6	8
Y_i	2	11	28	40

$$\varphi_{(x)} = a_0\varphi_0(x) + a_1\varphi_1(x)$$
$$\varphi_{-1} = 0;\ b_0 = 0;\ \varphi_0 = 1$$

$$\varphi_1 = (x - C_0) * \varphi_0 - b_0 * \varphi_{-1} = (x - C_0) * \varphi_0 = x - 5$$

$$C_0 = \frac{(x * \varphi_{0;}\varphi_0)}{(\varphi_0;\varphi_0)} = \frac{(x;1)}{(1;1)} = \frac{20}{4} = 5$$
$$a_0 = \frac{(f;\varphi_0)}{(\varphi_0;\varphi_0)} = \frac{(f;1)}{(1;1)} = \frac{81}{4}$$

$$a_1 = \frac{(f;\varphi_1)}{(\varphi_1;\varphi_1)} = \frac{(f;x-5)}{(x-5;x-5)} = \frac{2*(2-5)+11(4-5)+28*(6-5)+40*(8-5)}{(2-5)^2+(4-5)^2+(6-5)^2+(8-5)^2} = \frac{131}{20}$$

$$\varphi(x) = \frac{81}{4} * 1 + \frac{131}{20}(x-5) = -12{,}5 + 6{,}55x$$

Approximation using the trigonometrical polynomials

- periodical function
- using bigger numbers of TRIG. SERIES → approximation of every continuous function in closed interval
  → limit case → construction of the Fourier series
- oscilating systems, vibrations → periodical character → functions sin x, cos x
- per. Functions $f(t) = f * (t + T)$ T period (the smallest from possible)

$f(x) = A_0 + C \cdot \sin(\omega_0 t + \varphi)$ → substitution $x = \omega_0 t$

$f(x) = A_0 + C \cdot \sin(x + \varphi)$ → $\sin(x + \varphi) = \sin x \cos\varphi + \cos x \sin\varphi$

$f(x) = A_0 + C \cdot \sin x \cos\varphi + \mathrm{C} \cdot \cos x \sin\varphi$ → $A_1 = C \cdot \sin\varphi$ $B_1 = C \cdot \cos\varphi$

$f(x) = A_0 + A_1 \cos x + B_1 \sin x + r$ → r=residuum

$[x_i, y_i]$; i=1,2,…,N LSA we will minimize central quadratic mistake

$$N[E_2(f)]^2 = \sum_{i=0}^{N}[y_i - (A_0 + A_1\cos x_i + B_1 \sin x_i)]^2$$

$$\frac{\partial N}{A_0} = 2\sum_{i=0}^{N}\{[y_i - (A_0 + A_1\cos x_i + B_1\sin x_i)] \cdot (-1)\} = 0$$

$$\frac{\partial N}{A_1} = 2\sum_{i=\frac{1}{N}}^{N}\{[y_i - (A_0 + A_1\cos x_i + B_1 \sin x_i)]\cdot(-\cos x_i)\} = 0$$

$$\frac{\partial N}{B_1} = 2\sum_{i=0}^{N}\{[y_i - (A_0 + A_1\cos x_i + B_1 \sin x_i)]\cdot(-\sin x_i)\} = 0$$

System after simplification:

$$\begin{bmatrix} N & \sum_{i=0}^{N}\cos x_i & \sum_{i=0}^{N}\sin x_i \\ \sum_{i=0}^{N}\cos x_i & \sum_{i=0}^{N}\cos^2 x_i & \sum_{i=0}^{N}\sin x_i\cdot\cos x_i \\ \sum_{i=0}^{N}\sin x_i & \sum_{i=0}^{N}\cos x_i\cdot\sin x_i & \sum_{i=0}^{N}\sin^2 x_i \end{bmatrix}\cdot\begin{bmatrix} A_0 \\ A_1 \\ B_1\end{bmatrix} = \begin{bmatrix}\sum_{i=0}^{N} y_i \\ \sum_{i=0}^{N} y_i\cdot\cos x_i \\ \sum_{i=0}^{N} y_i\cdot\sin x_i\end{bmatrix}$$

considerations: equidistant points Xi $<-\pi; \pi>$ = sums L side

$X_I = -\pi + \frac{2\pi_I}{N}$ i=0,1,…. N

1) $\sum_{i=j}^{N} sinx_i = 0 \rightarrow \sum_{i=j}^{N} cosx_i = \sum_{i=j}^{N} sinx, \cos x = \frac{1}{2} sin2x_i$
2) $\sum_{i=1}^{N} cos^2x_i = \sum_{i=j}^{N}\frac{1+cos2x_i}{2} = \frac{N}{2} + \frac{1}{2}\sum_{i=1}^{N} cos2x_i = \frac{N}{2}$

$$\sum_{I=1}^{N} sin^2x_i = \sum_{i=1}^{N}(1 - cos^2x_i) = N - \frac{N}{2} = \frac{N}{2}$$

$$\Rightarrow \text{System} \begin{bmatrix} N & 0 & 0 \\ 0 & \frac{N}{2} & 0 \\ 0 & 0 & \frac{N}{2}\end{bmatrix}\begin{bmatrix}A_0 \\ A_1 \\ A_1\end{bmatrix} = \begin{bmatrix}\sum_0^N y_i \\ \sum_{I=0}^{N} y_i cosx_i \\ \sum_{i=0}^{N} sinx_0\end{bmatrix}$$

Regular matrix ⇒ inverse

Inversion matrix

$$\begin{matrix} N \\ \frac{N}{2} \\ \frac{N}{2}\end{matrix}\left(\begin{matrix} N & 0 & 0 \\ 0 & \frac{N}{2} & 0 \\ 0 & 0 & \frac{N}{2}\end{matrix}\left|\begin{matrix}1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1\end{matrix}\right.\right) \approx \left(\begin{matrix}1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1\end{matrix}\left|\begin{matrix}\frac{1}{2} & 0 & 0 \\ 0 & \frac{2}{N} & 0 \\ 0 & 0 & \frac{2}{N}\end{matrix}\right.\right) \rightarrow \begin{bmatrix}\frac{1}{2} & 0 & 0 \\ 0 & \frac{2}{N} & 0 \\ 0 & 0 & \frac{2}{N}\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} A_0 \\ A_1 \\ A_1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{2}{N} & 0 \\ 0 & 0 & \frac{2}{N} \end{bmatrix} * \begin{bmatrix} \sum_{i=0}^{N} x_i \\ \sum_{i=0}^{N} y_i cosx_i \\ \sum_{i=0}^{N} y_i sinx_i \end{bmatrix} \Rightarrow \begin{matrix} A_0 = \frac{1}{2}\sum_{i=0}^{N} y_i \\ A_1 = \frac{2}{N}\sum_{i=0}^{N} y_i \, cosx_i \\ B_1 = \frac{2}{N}\sum_{I=0}^{N} y_i sin\, x_i \end{matrix}$$

Generalization: Periodical function, period 2π given by N+1 equidistant points $[x_i; y_i][x_i y_i]_{i=0}^{N}$

xi=-π+$\frac{2\pi * i}{N}$ $we\ can\ approximate\ by\ T\ polynomial$

$$T_M(x) = A_0 + A_1 cosx + B_1 sinx + \ldots\ldots + A_M \cos(M * x) + B_M \sin(M * x)$$

Coefficients $A_0 = \frac{1}{N}\sum_{i=1}^{N} y_i$

$$A_j = \frac{2}{N}\sum_{i=1}^{N} y_i \cos(j\, x_i)$$

where j=1,2,……, M.

$$B_j = \frac{2}{N}\sum_{i=1}^{N} y_i \sin(j\, x_i)$$

M is the degree of T polynomial and satisfy: N>2*M+1

⇒Numerical approximation of the coefficients of the Fourier series

$$A_j = \int_{-\pi}^{\pi} f_{(x)} * \cos(jx)\, dx \ \rightarrow j = 0,1, \ldots\ldots M$$

$$B_j = \int_{-\pi}^{\pi} f_{(x)} * \sin(jx)\, dx \ \rightarrow j = 1,2, \ldots\ldots M$$