PM3DP – PM3DK – ukázková PP zk

TEORIE

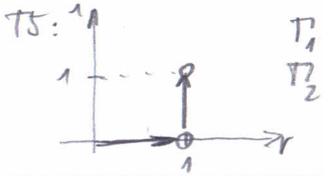
T1: $[r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta)$

T2: $\lim\limits_{z\to i\frac{\sqrt{3}}{2}} \dfrac{4z^2+3}{2z-i\sqrt{3}} = \lim\limits_{z\to i\frac{\sqrt{3}}{2}} \dfrac{4\left(z+i\frac{\sqrt{3}}{2}\right)\cancel{\left(z-i\frac{\sqrt{3}}{2}\right)}}{2\cancel{\left(z-i\frac{\sqrt{3}}{2}\right)}} = 2\left(i\frac{\sqrt{3}}{2}+i\frac{\sqrt{3}}{2}\right) = 2\sqrt{3}i$

T3: $(-1)^i = e^{i\ln(-1)} = e^{i(\ln 1 + i\pi)} = e^{i^2\pi} = e^{-\pi} = \frac{1}{e^{\pi}} \doteq 0{,}0432$

T4: $\frac{\partial u}{\partial x}(A) = \frac{\partial v}{\partial y}(A) \wedge \frac{\partial u}{\partial y}(A) = -\frac{\partial v}{\partial x}(A)$

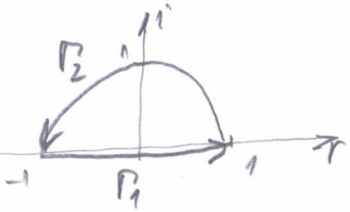
T5:



$\Gamma_1:\ z = t \ ;\ t \in \langle 0;1\rangle$

$\Gamma_2:\ z = 1 + it \ ;\ t \in \langle 0;1\rangle$

② $\displaystyle\int_\Gamma \frac{z}{\bar z}\,dz = 2 + \left(-\frac{2}{3}\right) = \frac{4}{3}$



$\Gamma_1: z(t) = t\ ;\ t \in \langle -1;1\rangle \quad \frac{z}{\bar z} = \frac{t}{t} = 1;\ dz = dt$

$\displaystyle\int_{\Gamma_1} \frac{z}{\bar z}\,dz = \int_{-1}^{1} 1\,dt = [t]_{-1}^{1} = [1-(-1)] = 2$

$\Gamma_2: z(t) = e^{it}\ ;\ t \in \langle 0;\pi\rangle \quad \frac{z}{\bar z} = \frac{e^{it}}{e^{-it}} = e^{2it}\ ;\ dz = ie^{it}\,dt$

$\displaystyle\int_{\Gamma_2} \frac{z}{\bar z}\,dz = \int_0^{\pi} e^{2it}\cdot i e^{it}\,dt = i\int_0^{\pi} e^{3it}\,dt = i\cdot\frac{1}{3i}\left[e^{3it}\right]_0^{\pi} \ominus$

$\ominus\ \frac{1}{3}\left[e^{3i\pi} - e^{0}\right] = \frac{1}{3}(\underbrace{\cos 3\pi}_{-1} + i\underbrace{\sin 3\pi}_{0} - 1) = -\frac{2}{3}$

① $f(z) = \sqrt[4]{z^2-2z-1}\ ;\ z = 2+i$

$f(2+i) = \sqrt[4]{(2+i)^2 - 2(2+i) - 1} = \sqrt[4]{4+4i+i^2-4-2i-1} = \sqrt[4]{-2+2i} = \sqrt[4]{2\sqrt{2}}\left[\cos\frac{\frac{3\pi}{4}+2k\pi}{4} + i\sin\frac{\frac{3\pi}{4}+2k\pi}{4}\right] \ominus$

$z = -2+2i$

$|z| = 2\sqrt{2}$

$\left.\begin{array}{l}\cos\varphi = -\frac{\sqrt{2}}{2}\\ \sin\varphi = \frac{\sqrt{2}}{2}\end{array}\right\}\varphi = \frac{3\pi}{4} + 2k\pi$

$\ominus\ (k=0) \to \sqrt[4]{2\sqrt{2}}\left(\cos\frac{3\pi}{16} + i\sin\frac{3\pi}{16}\right) = \sqrt[4]{2\sqrt{2}}\cdot e^{\frac{3\pi}{16}i}$

$(k=1) \to \sqrt[4]{2\sqrt{2}}\left(\cos\frac{11\pi}{16} + i\sin\frac{11\pi}{16}\right) = \sqrt[4]{2\sqrt{2}}\, e^{\frac{11\pi}{16}i}$

$(k=2) \to \sqrt[4]{2\sqrt{2}}\left(\cos\frac{19\pi}{16} + i\sin\frac{19\pi}{16}\right) = \sqrt[4]{2\sqrt{2}}\, e^{\frac{19\pi}{16}i}$

$(k=3) \to \sqrt[4]{2\sqrt{2}}\left(\cos\frac{27\pi}{16} + i\sin\frac{27\pi}{16}\right) = \sqrt[4]{2\sqrt{2}}\, e^{\frac{27\pi}{16}i}$

③ $y'' - 3y' + 2y = e^{-4t}\ ;\ y(0) = 1\ ;\ y'(0) = 5$

$\mathcal{L}\{y''\} - 3\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = \mathcal{L}\{e^{-4t}\}$

$p^2F(p) - p - 5 - 3[pF(p) - 1] + 2F(p) = \frac{1}{p+4}$

$F(p)[p^2 - 3p + 2] = \frac{1}{p+4} + p + 2 = \frac{p^2+6p+9}{p+4} = \frac{(p+3)^2}{p+4}$

$F(p) = \frac{(p+3)^3}{(p+4)(p^2-3p+2)} = \frac{p^2+6p+9}{(p+4)(p-2)(p-1)} = \frac{A}{p+4} + \frac{B}{p-1} + \frac{C}{p-2}$

$A = \left.\frac{p^2+6p+9}{(p-1)(p-2)}\right|_{p=-4} = \frac{1}{(-5)(-6)} = \frac{1}{30} \qquad B = \left.\frac{p^2+6p+9}{(p+4)(p-2)}\right|_{p=1} = \frac{16}{5(-1)} = -\frac{16}{5}$

$C = \left.\frac{p^2+6p+9}{(p+4)(p-1)}\right|_{p=2} = \frac{25}{6\cdot 1} = \frac{25}{6}$

$y(t) = \mathcal{L}^{-1}\left\{\frac{\frac{1}{30}}{p+4} - \frac{\frac{16}{5}}{p-1} + \frac{\frac{25}{6}}{p-2}\right\} = \frac{1}{30}\mathcal{L}^{-1}\left\{\frac{1}{p+4}\right\} - \frac{16}{5}\mathcal{L}^{-1}\left\{\frac{1}{p-1}\right\} + \frac{25}{6}\mathcal{L}^{-1}\left\{\frac{1}{p-2}\right\} \Rightarrow$

$\Rightarrow y(t) = \frac{1}{30}e^{-4t} - \frac{16}{5}e^{t} + \frac{25}{6}e^{2t}$

④ $y'' - 3y' + 2y = e^{-4t}$; $y(0) = 1$; $y'(0) = 5$

$(y = y_1)' \to y' = y_1' = y_2$ DOSAZ: $y_2' - 3y_2 + 2y_1 = e^{-4t}$

$(y' = y_2)' \to y'' = y_2'$ $y_2' = -2y_1 + 3y_2 + e^{-4t}$

$$y_1' = y_2$$
$$y_2' = -2y_1 + 3y_2 + e^{-4t} \qquad y' = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix} y + \begin{pmatrix} 0 \\ e^{-4t} \end{pmatrix}$$

$$y_H = C_1 \begin{pmatrix} e^t \\ e^t \end{pmatrix} + C_2 \begin{pmatrix} e^{2t} \\ 2e^{2t} \end{pmatrix} \Rightarrow y = C_1(x) \begin{pmatrix} e^t \\ e^t \end{pmatrix} + C_2(x) \begin{pmatrix} e^{2t} \\ 2e^{2t} \end{pmatrix}$$

$$\Rightarrow \text{Soustava: } \begin{pmatrix} e^t & e^{2t} \\ e^t & 2e^{2t} \end{pmatrix} \begin{pmatrix} C_1'(x) \\ C_2'(x) \end{pmatrix} = \begin{pmatrix} 0 \\ e^{-4t} \end{pmatrix}$$

$$C_1'(x) = \frac{\begin{vmatrix} 0 & e^{2t} \\ e^{-4t} & 2e^{2t} \end{vmatrix}}{\begin{vmatrix} e^t & e^{2t} \\ e^t & 2e^{2t} \end{vmatrix}} = \frac{-e^{-2t}}{e^{3t}} = -e^{-5t} \qquad C_1(t) = -\int e^{-5t}\,dt = \frac{1}{5} e^{-5t} + C_1$$

$$C_2'(x) = \frac{\begin{vmatrix} e^t & 0 \\ e^t & e^{-4t} \end{vmatrix}}{e^{3t}} = e^{-6t} \qquad C_2(x) = \int e^{-6t}\,dt = -\frac{1}{6} e^{-6t} + C_2$$

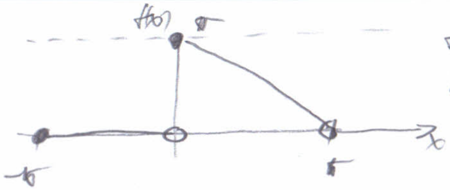
$$\text{OŘ: } \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \left(\frac{1}{5} e^{-5t} + C_1\right) \begin{pmatrix} e^t \\ e^t \end{pmatrix} + \left(-\frac{1}{6} e^{-6t} + C_2\right) \begin{pmatrix} e^{2t} \\ 2e^{2t} \end{pmatrix} = \begin{pmatrix} \frac{1}{30} e^{-4t} + C_1 e^t + C_2 e^{2t} \\ -\frac{2}{15} e^{-4t} + C_1 e^t + 2C_2 e^{2t} \end{pmatrix}$$

$$\text{PŘ: } \begin{pmatrix} 1 \\ 5 \end{pmatrix} = \begin{pmatrix} \frac{1}{30} e^0 + C_1 e^0 + C_2 e^0 \\ -\frac{2}{15} e^0 + C_1 e^0 + 2C_2 e^0 \end{pmatrix} \Rightarrow \begin{aligned} C_1 + C_2 &= \frac{29}{30} \\ C_1 + 2C_2 &= \frac{77}{15} \end{aligned}$$

$$\text{II} - \text{I}: \quad C_2 = \frac{25}{6}$$
$$C_1 = \frac{29}{30} - \frac{25}{6} = -\frac{16}{5}$$

$$y(t) = y_1 = \frac{1}{30} e^{-4t} - \frac{16}{5} e^t + \frac{25}{6} e^{2t} \quad (\text{viz př. 3})$$

⑤ $f(x) = \begin{cases} 0 & -\pi \le x < 0 \\ \pi - x & 0 \le x < \pi \end{cases}$



$f(x)$ je po částech spojitá, není S ani L

$T = 2\pi \quad \omega = \frac{2\pi}{T} = 1 \quad b_n = \frac{1}{n}$

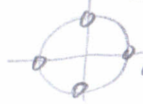
$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi} \int_{-\pi}^{0} 0\,dx + \frac{1}{2\pi} \int_0^{\pi} (\pi - x)\,dx = 0 + \frac{1}{2\pi} \left[\pi x - \frac{x^2}{2}\right]_0^{\pi} = \frac{1}{2\pi}\left(\pi^2 - \frac{\pi^2}{2}\right) = \frac{\pi}{4}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\,dx = \frac{1}{\pi} \int_0^{\pi} (\pi - x) \cos nx\,dx = \frac{1}{\pi} \left[\frac{\pi - x}{n} \sin nx - \frac{1}{n^2} \cos nx\right]_0^{\pi} \ominus$$

$$\int (\pi - x) \cos nx\,dx = \frac{\pi - x}{n} \sin nx + \frac{1}{n} \int \sin nx\,dx = \frac{\pi - x}{n} \sin nx - \frac{1}{n^2} \cos nx$$

$u = \pi - x \quad u' = -1$

$v' = \cos nx \quad v = \frac{1}{n} \sin nx$



$\sin n\pi = 0$

$\cos n\pi = (-1)^n$

$$\ominus \frac{1}{\pi}\left[\left(0 - \frac{1}{n^2} \overbrace{\cos n\pi}^{(-1)^n}\right) - \left(\frac{\pi}{n} \sin 0 - \frac{1}{n^2} \overbrace{\cos 0}^{1}\right)\right] = \frac{1}{\pi}\left[-\frac{1}{n^2}(-1)^n + \frac{1}{n^2}\right] \ominus$$

$$\ominus \frac{1}{\pi n^2}\left[1 - (-1)^n\right] = \begin{cases} 0 & n \text{ sudé } n = 2k \\ \frac{2}{\pi n^2} & n \text{ liché } n = 2k-1 \end{cases}$$

$$\underset{x \in \langle -\pi, \pi \rangle}{f(x)} = \begin{cases} \frac{\pi}{2}, & x = 0 \\ \frac{\pi}{4} + \sum_{n=1}^{\infty} \left\{\frac{1}{\pi n^2}\left[1 - (-1)^n\right] \cos nx + \frac{1}{n} \sin nx\right\} = \frac{\pi}{4} + \underbrace{\left(\frac{2}{\pi} \cos x + \sin x\right)}_{n=1} \oplus \end{cases}$$

$$\oplus \underbrace{\left(0 + \frac{1}{2} \sin 2x\right)}_{n=2} + \underbrace{\left(\frac{2}{9\pi} \cos 3x + \frac{1}{3} \sin 3x\right)}_{n=3} + \underbrace{\left(0 + \frac{1}{4} \sin 4x\right)}_{n=4} + \underbrace{\left(\frac{2}{25\pi} \cos 5x + \frac{1}{5} \sin 5x\right)}_{n=5} + \dots$$