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MINISTERSTVO ŠKOLSTVÍ, MLÁDEŽE A TĚLOVÝCHOVY

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The theory of the Residuals

The Laurent Series (LS)

Let Ω is the inter-circular region with the center in the point z_0. If the complex function $f(z)$ is the holomorphic in the regionΩ, subsequently for each $z \in \Omega$ is possible to express the function $f(z)$ using the Laurent series: $f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$,

Where the constants a_n are given using the formulas: $a_n = \frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{(z-z_0)^{n+1}} dz$ and Γ is any circle laying in the inter-circular region Ω.

- The part $\sum_{n=-\infty}^{-1} a_n (z - z_0)^n$ is called the main part of the LS
- The Taylor Series is the specific case of the LS (main part is equal to 0)
- For the expansion of the rational complex function in LS we use the formula for the sum of the geometric series (it is simplify the curve integral)
    - the geometric series: a sequence $\{a_n\}_{n=1}^{\infty} = \{a_0 q^n\}_{n=0}^{\infty}$ is called the geometric sequence. If $|q| \geq 1$ GS is the divergent, if $|q| < 1$ GS is convergent and $\sum_{n=0}^{\infty} a_0 q^n = \frac{a_0}{1-q}$.

The singular points of the complex function, The Residual theorem

Lets the complex function $f(z)$ is holomorphic in a certain surrounding of the point z_0 with the the exception of the point z_0. Then z_0 is called the *singular point* of the function $f(z)$.

The classification of the singular points:

- Removable singularity
    - The main part of the LS in the point z_0 is equal to 0
    - $\lim_{z \to z_0} f(z) = a_0$ is finity
- Significant singularity
    - The main part of the LS in the point z_0 consists from the infinity nonzero members
    - $\lim_{z \to z_0} f(z)$ does not exists
- The m-th order pole
    - The main part of the LS in the point z_0 consists from the finity nonzero members
    - for $a_n = 0$ and for $n < -m$; $a_{-m} \neq 0$ is true that $\lim_{z \to z_0} f(z) = \infty$

- For determining the order of the pole we use the limit $\lim_{z \to z_0} (z - z_0) f(z) = a_{-m}$, which always have to be finite and nonzero.
- The coefficient a_{-1} of the LS of the functions $f(z)$ in the point z_0 is called the reziduum of the function $f(z)$ in the point z_0. We note $a_{-1} = \operatorname*{res}_{z=z_0} f(z)$
- The formula for the calculation of the residuum for the 1-st order pole is $\operatorname*{res}_{z=z_0} f(z) = \lim_{z \to z_0} (z - z_0) \cdot f(z)$
- The formula for the calculation of the residuum for the m-th order pole is
$$\operatorname*{res}_{z=z_0} f(z) = \frac{1}{(m-1)!} \lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}} [(z - z_0)^m \cdot f(z)]$$
- Lets $f(z) = \frac{\varphi(z)}{\psi(z)}$, where the functions $\varphi(z)$ and $\psi(z)$ are the holomorphic in the point z_0, $\varphi(z_0) \neq 0, \psi(z_0) = 0,\ \psi'(z_0) \neq 0$, then the function $f(z)$ has in the point z_0 the 1-st order pole and $\operatorname*{res}_{z=z_0} f(z) = \frac{\varphi(z_0)}{\psi'(z_0)}$.

The residual theorem

Let's the complex function $f(z)$ holomorphic inside and at the simply closed and positive oriented curve Γ with exeption the poles $z_1, z_2, \cdots, z_n$. Inside the curve Γ is:

$$\int_\Gamma f(z)dz = 2\pi i \cdot \sum_{k=1}^{n} \operatorname*{res}_{z=z_k} f(z).$$

So we only add the residuals in the poles inside the curve Γ.