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First-order System of Ordinary Differential Equation (1-OSODE)

Second-order homogeneous linear DE (2-OHOD) – summary

$$ay'' + by' + cy = 0; a, b, c \in R; a \neq 0$$

The roots of auxiliary equation (AE) $ar^2 + br + c = 0$	General solution
r_1, r_2 real various	$y = C_1e^{r_1x} + C_2e^{r_2x}$
$r_1 = r_2 = r$	$y = C_1e^{rx} + C_2xe^{rx}$
r_1, r_2 complex conjugate $\alpha \pm \beta i$	$y = C_1e^{\alpha x}\cos\beta x + C_2e^{\alpha x}\sin\beta x$

Non-homogeneous linear second–order differential equation (NHL2ODE)

$$ay'' + by' + cy = G(x); G(x) \neq 0;\ a, b, c \in R; a \neq 0$$

The general solution consists from homogeneous solution y_h and particular solution y_p

$$y = y_h + y_p$$

y_h we determine see above

y_p we determine using the indefinite coefficients method by estimating according to the shape of $G(x)$

- $G(x)$ is the n-th order polynomial, (e.g. 2-nd order polynomial) → $y_p = Ax^2 + Bx + C$
- $G(x)$ is in the shape of $Ce^{kx}; C, k \in R,$→ $y_p = Ae^{kx}$
- $G(x)$ is in the shape of $C\cos kx\ \ or\ C\sin kx\ \ ; C, k \in R,$→ $y_p = A\cos kx + B\sin kx$
- $G(x)$ is the product of the foregoing shapes → y_p is the product of the estimations (e.g. $G(x) = x \cdot \cos 3x$ → $y_p = (Ax + B)\cos 3x + (Cx + D)\sin 3x$
- $G(x)$ is the sum of the foregoing shapes → Principle of the Superposition (e.g. $G(x) = G_1(x) + G_2(x)$ → $y_p = y_{p1} + y_{p2}$
- Generalization $G(x) = e^{ax}[P_n(x)\cos bx + Q_m(x)\sin bx]$, m, n degree $P_n(x)$ and $Q_m(x)$ $p = max(m; n)$ is the degree of $R_p(x)$ or rather. $S_p(x)$
  - $a + bi$ is not the root of AE $y_p = e^{ax}\left[R_p(x)\cos bx + S_p(x)\sin bx\right]$
  - $a + bi$ is the k-multiple root of AE $y_p = x^k e^{ax}\left[R_p(x)\cos bx + S_p(x)\sin bx\right]$

First-order System of the Ordinary Differential Equations (1OSODR)

Basic concepts

- Engineering practise – e.g. Mechanics tasks
- The first-order system of the n differential equations
  $y_1' = f_1(x; y_1; \cdots; y_n)$
  $y_2' = f_2(x; y_1; \cdots; y_n)$, $\quad [A]$

  $y_n' = f_n(x; y_1; \cdots; y_n)$
  where the functions $f_k(k = 1; \cdots; n)$ are defined on $(n+1)$-dimensional area $\Omega \in \mathrm{R}^{n+1}$, we call the First-order System of the Ordinary Differential Equations (1OSODR)
- *The solution* of the 1OSODR we call every group of n functions in the shape $y_1(x), y_2(x), \cdots, y_n(x)$, which are the continuous smooth differentiable in any interval $I \subset \Omega$ and which are convenient for 1OSODR for all points $x \in I$
- The task of finding the solution of 1OSODR which satisfied n initial conditions $y_1(x_0) = b_1, y_2(x_0) = b_2, \cdots, y_n(x_0) = b_n$, where $[x_0; b_1; b_2; \cdots; b_n] \in \Omega$, is arbitrary but tightly elected point, we call the *initial problem* for 1OSODR
- *The general solution* of 1OSODR is every group of n functions which depend to n general parameters $C_1; C_2; \cdots; C_n$ such suitable election of the constants we get the solution of the each initial problem
- *The particular solution* of the 1OSODR we called each solution, which we determine from the general solution using the tight election of the constants $C_1; C_2; \cdots; C_n$.

Convert NHL2ODE into 1OSODR

- All solid-body mechanics and rigid bodies (including related fields) are built on the second Newton's law, which is generally in the form of a system of three nonlinear NHL2ODE coordinate:

$$
\begin{aligned}
x''(t) &= f\big(t; x(t); y(t); z(t); x'(t); y'(t); z'(t)\big) \\
y''(t) &= g\big(t; x(t); y(t); z(t); x'(t); y'(t); z'(t)\big), \quad [B] \\
z''(t) &= h\big(t; x(t); y(t); z(t); x'(t); y'(t); z'(t)\big)
\end{aligned}
$$

  Where $x''(t)$ the second derivation of the path by time is, $x'(t)$ is the first derivation of the path by time and f, g, h are given functions.

- The initial conditions (characterizing the initial position and initial velocity) are in the form of:
  $x(t_0) = x_0; y(t_0) = y_0; z(t_0) = z_0; x'(t_0) = u_0; y'(t_0) = v_0; z'(t_0) = w_0, \quad [C]$
  where $x_0; y_0; z_0; u_0; v_0; w_0$ are given numbers.
- Problem $[B] + [C]$ we transform in problem $[A]$ using this way:
  - We put: $u(t) = x'(t);\ v(t) = y'(t);\ w(t) = z'(t) \quad [D]$
  - We rewrite the equations $[B]$ in the shape:
    $u'(t) = X\big(t; x(t); y(t); z(t); u(t); v(t); w(t)\big)$
    $y''(t) = Y\big(t; x(t); y(t); z(t); u(t); v(t); w(t)\big), \quad [E]$
    $z''(t) = Z\big(t; x(t); y(t); z(t); u(t); v(t); w(t)\big)$
  - We add the relationships $[D]$, into these three equations:
    $x'(t) = u(t);\ y'(t) = v(t);\ z'(t) = w(t) \quad [F]$
  - The initial conditions we rewrite:
    $x(t_0) = x_0; y(t_0) = y_0; z(t_0) = z_0; u(t_0) = u_0; v(t_0) = v_0; w(t_0) = w_0$
- The system $[E] + [F]$ forms the system of 1OSODR for six unknown functions

Example:

Initial-value problem $y^{(IV)} - 2y'' + y = 0; y(0) = 0, y'(0) = 1, y''(0) = 0, y^{(III)}(0) = -1$ we rewrite into the na initial-value problem of 1OSODR.

- We get: $y_1 = y;\ y_2 = y';\ y_3 = y'';\ y_4 = y^{(III)};$
- We use the derivation: $y_1' = y' = y_2;\ y_2' = y'' = y_3;\ y_3' = y^{(III)} = y_4;$
- After we substitute into the original equation we get the system 1OSODR:

$$
\begin{aligned}
y_1' &= y_2 \\
y_2' &= y_3 \\
y_3' &= y_4 \\
y_4' &= -y_1 + 2y_3
\end{aligned}
$$

- With initial conditions: $y_1(0) = 0;\ y_2(0) = 1;\ y_3(0) = 0;\ y_4(0) = -1;$
- The generalization of this algorithm we can convert every n-th order DE into the system of n 1-th order DE.

The system of linear ordinary 1-st order DE (1OLODE)

1OLODE is the system of DE in the shape:

$$\begin{aligned} y_1' &= a_{11}(x)y_1 + a_{12}(x)y_2 + \cdots a_{1n}(x)y_n + f_1(x) \\ y_2' &= a_{21}(x)y_1 + a_{22}(x)y_2 + \cdots a_{2n}(x)y_n + f_2(x) \\ &\cdots \\ y_n' &= a_{n1}(x)y_1 + a_{n2}(x)y_2 + \cdots a_{nn}(x)y_n + f_n(x) \end{aligned}, \qquad [G]$$

Where all coefficients (functions) $a_{ij}(x)$ and function $f_k(x)$ are continuos in interval I.

- If $f_k(x) = 0, \forall x \in I$ and $k = 1,2,\cdots,n$ we are talking about a homogenous system 1OLODE
- If $f_k(x) \neq 0$ for some $x \in I$ and $k = 1,2,\cdots,n$ we are talking about a nonhomogenous system 1OLODE
- If the system has a constant coefficients, we will write instead of $a_{ij}(x)$ only a_{ij}.
- Exact solution has only the systems whit constant coefficients

The matrix notation of 1OLODE

Only constant coefficients

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}.$$

Columns: $y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}, y' = \begin{pmatrix} y_1' \\ \vdots \\ y_n' \end{pmatrix}, f(x) = \begin{pmatrix} f_1(x) \\ \vdots \\ f_n(x) \end{pmatrix}.$

The system $[G]$ is in the shape:

$$y' = A \cdot y + f(x).$$

If we implement the constant vector $b = (b_1; b_1; \cdots; b_n)^T$, where $b_i, i = 1,\cdots,n$ are the right sides of the initial conditions, we can write the initial conditions in the shape $y(x_0) = b$.

General homogeneous system 1OLODE

$$y' = A(x) \cdot y \qquad (H)$$

Linearity of the solution space

If the vectors $y_1, y_2, \cdots, y_k$ are the solutions of the system $y' = A(x) \cdot y$ at interval I, then their linear combinations $y = C_1 y_1 + C_2 y_2 + \cdots + C_k y_k$ are the solutions of this system at interval I.

Verification using substitution to H:
$y' = A(x) \cdot y \ \rightarrow (C_1 y_1 + C_2 y_2 + \cdots + C_k y_k)' = A(x)(C_1 y_1 + C_2 y_2 + \cdots + C_k y_k)$
we derivative $\ \rightarrow C_1 y_1' + C_2 y_2' + \cdots + C_k y_k' = A(x)(C_1 y_1 + C_2 y_2 + \cdots + C_k y_k) \rightarrow$

$$\rightarrow y_1' = A(x) y_1 \cdots y_k' = A(x) y_k$$

The fundamental system

If the vectors $y_1, y_2, \cdots, y_n$ are the linear independent at interval I and if they are the solutions of the system H, then we can say, that they constitute the *fundamental system* of the solution of the system H at interval I.

The fundamental matrix

Let the vectors $y_1, y_2, \cdots, y_n$ are the solution of the system H. These vectors are the linear independent at iterval I just when the matrix

$Y = Y(x) = [y_1, y_2, \cdots, y_n] = \begin{pmatrix} y_{11} & y_{12} & \cdots & y_{1n} \\ y_{21} & y_{22} & \cdots & y_{2n} \\ \cdots & \iddots & \ddots & \cdots \\ y_{n1} & y_{n2} & \cdots & y_{nn} \end{pmatrix}$, its columns are formed by vectors $y_1, y_2, \cdots, y_n$ has at interval I nenzero determinant. Matrix is called the *fundamental matrix* of the system.

The structure of the general solution

If the vectors $y_1, y_2, \cdots, y_n$ constitute the fundamental system of the solution of the homogenous system $y' = A(x) \cdot y$ at the interval I, then each solution y of this system is possible to express uniquely in the shape
$y = C_1 y_1 + C_2 y_2 + \cdots + C_n y_n$, where $C_i, i = 1, \cdots, n$ are the suitable constants.

If we put $C = (C_1, C_2, \cdots, C_n)^T$, then $y = Y \cdot C = C^T \cdot Y^T$, where Y is the fundamental matrix.

$y = C_1 y_1 + C_2 y_2 + \cdots + C_n y_n$ is the expression of the general solution of the system $y' = A(x) \cdot y$.

The homogeneous system of 1OLODE with constant coefficients

The shape: $y' = A \cdot y$ $\qquad (G)$, where A is a constant matrix

- For determining the general solution GS of the system we need to find n linearly independent solutions of this system
- **The Euler method of eigenvalues** leads to find these n linearly independent solutions
- The construction of the fundamental system of the solution:
    - We find the particular solution $y_p = e^{\lambda x} \cdot h = e^{\lambda x}(h_1, h_2, \cdots, h_n)^T$, where we neet to determine unknowns $\lambda \in C$ a $h = (h_1, h_2, \cdots, h_n)^T \in C^n$
    - Vector $h = (h_1, h_2, \cdots, h_n)^T$ is nonzero (at least one component is nonzero)
    - Using substitution into (G) we get: $y_p' = \lambda \cdot e^{\lambda x} \cdot h = A \cdot \left(e^{\lambda x} \cdot h\right) = e^{\lambda x} \cdot A \cdot h$
    - In equation $\lambda \cdot e^{\lambda x} \cdot h = e^{\lambda x} \cdot A \cdot h$ we reduce by $e^{\lambda x} \ \rightarrow \ A \cdot h = \lambda \cdot h$
    - The numbers λ for which has the equation $A \cdot h = \lambda \cdot h$ nonzero solution $h \neq 0$ (at least one component of vector h is nonzero) we call the eigenvalues of the matrix A
    - Constant nonzero vectos h, which for given eigenvalue λ satisfied the equation $A \cdot h = \lambda \cdot h$ are called the eigenvectors of matrix A appropriate to eigenvalue λ
- The procedure for determining eigenvalues of square matrix A
    - Using the formula $h = E \cdot h$, where E is the unit matrix (e.g: $E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
    - $A \cdot h = \lambda E \cdot h \rightarrow (A - \lambda E) \cdot h = o$, where $o = (0,0,\cdots,0)^T$
    - It is the homogeneous system of equations, which has infinity of solutions $h = (h_1, h_2, \cdots, h_n)^T$
    - The determinant of the system we put to 0 $\rightarrow |A - \lambda E| = 0$
    - We get the auxiliary equation of the system $(G) \rightarrow$
      $$\begin{vmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \cdots & \iddots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{vmatrix} = 0$$
    - This is the n-th order algebraic equation for unknown λ, which has n complex roots inside its multiplicity
    - E.g. for $n = 2$ it is the quadratic equation in shape $\begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix} = 0$ which we need to solve.
- The algorithm of solving **1OLODE** $y' = A \cdot y$
    - We find all eigenvalues λ of the matrix A
    - For each eigenvalue λ of the matrix A we find the eigenvector h
    - If all eigenvalues are the real and different each other, we find n linearly independent particular solution of the system $y' = A \cdot y$
    - If some of the eigenvalues λ are the multiple roots of the auxiliary equation or complex roots, we use the additional modifications (see examples lately)
- For $n = 2$ we find the two linearly independent unknown functions y_1, y_2
- GS is the linear combination of these solutions

The principle of the Superpozition

If we can in the nonhomogeneous system (H) the vector function $f(x)$ decomposed into sum of the shape

$$f(x) = f_1(x) + f_2(x) + \cdots + f_r(x)$$

And if we can to determine the particular solution $y_{pj}(x)$ of the equation $y' = A \cdot y + f_j(x)$, then $y_p(x) = y_{p1}(x) + y_{p2}(x) + \cdots + y_{pr}(x)$ is the particular solution of the system (H).

For finding of the GS (PS) of the nonhomogeneous system (H) we can use the methods of

- The variation of the constants
- The special shape of the right side (the method of the undetermined coefficients)

The method of the variation of the constants

We suppose that $y_1, y_2, \cdots, y_n$ are LIPS of the affiliated homogeneous system.

Let $Y(x)$ is the relevant fundamental matrix of the solution of the affiliated homogeneous system. Then the GS of this homogeneous system is in the shape:
$y_h(x) = C_1 y_1 + C_2 y_2 + \cdots + C_n y_n = Y(x) \cdot C$, where $C = (C_1; C_2; \cdots; C_n)^T$

GS $y(x)$ of the system $y' = A \cdot y + f(x) \quad (H)$ is

$$y(x) = C_1(x)y_1(x) + C_2(x)y_2(x) + \cdots + C_n(x)y_n(x) = Y(x) \cdot C(x) \qquad (I)$$

$C(x) = [C_1(x); C_2(x); \cdots; C_n(x)] \rightarrow$ the components $C(x)$ we need to find.

- We substitute (I) into (H):

$$Y'(x) \cdot C(x) + Y(x) \cdot C'(x) = A(x) \cdot Y(x) \cdot C(x) + f(x) \qquad (J)$$

- This is the system for unknowns $C_1'(x), C_2'(x), \cdots, C_n'(x)$
- The matrix of the system (J) is the fundamental matrix $Y(x)$ (determinant is nonzero)
- Using integration $\int C_j'(x)\, dx$ we get $C_j(x) \quad for\ j = 1 \cdots n \rightarrow$ GS y of thesystem $y' = A \cdot y + f(x) \quad (H)$

The note of the example 11:

- We have construct the complex solution $w = u + iv$
- For the finding of the two real solutions we use the knovledge:

If the homogeneous system $y' = Ay$ has a complex solution $y = u + iv$, where u, v are the real vectors then it has also the complex associated solution $\bar{y} = u - iv$ and solution u a v. Our system has also the real solutions
$u = e^{-6x}\begin{pmatrix} \cos x \\ \cos x - \sin x \end{pmatrix}$ and $v = e^{-6x}\begin{pmatrix} \sin x \\ \sin x + \cos x \end{pmatrix}$, they are the linearly independent.

The note of the example 12:

- The coefficients a, b in the formula $v = e^{-2x}\begin{pmatrix} ax + a_1 \\ bx + b_1 \end{pmatrix}$ we can choose the same as the components h_1, h_2 of the eigenvector h.
- We can to show that if λ is the double-multiple real eigenvalue of the matrix A such that the rank of the matrix $A - \lambda E$ is equal to one (rows are dependent and at least one is nonzero) then we can the particular solution u suppose in the usual shape $u = e^{\lambda x} h$ and solution v then in the shape $v = e^{\lambda x}(hx + \overline{h})$, where the vector $\overline{h}$ satisfies the condition of $(A - \lambda E)\overline{h} = h$.
- This algorithm we can not to use in the case that the matrix A has the double-multiple real eigenvalue λ such as the system is in the shape of:

$$y_1' = \lambda y_1$$
$$y_2' = \lambda y_2$$

  - This system we solve using the gradual integration
- For $n \geq 3$ this is the generalization of the shown cases.

The general nonhomogeneous system of 1OLODE

The shape: $y' = A \cdot y + f(x) \qquad (H)$,
where A is the constant matrix and $f(x)$ nonzero continuous vector function at interval I.

The structure or the general solution GS

The general solution of the system (H) we can write in the shape $y = y_h + y_p$, where y_h is the GS of the relevant homogeneous **1OLODE** and y_p is any particular solution of the original nonhomogeneous **1OLODE**.

- The exact solution we can find only for the systems with the constant coefficients

The method of the indeterminate coefficients

See example