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MINISTERSTVO ŠKOLSTVÍ, MLÁDEŽE A TĚLOVÝCHOVY

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The Laplace Transform

The Laplace Transform (LT)

Let the function $f: R \to R$ satisfies the followed conditions:

1. The function $f(t)$ is at least continuous by parts
2. $f(t) = 0$ for $t < 0$,
3. The function $f(t)$ is the exponential order, i.e. There exist the constants $M, s, t_0 \in R, t_0 > 0$, such for all $t \geq t_0$ holds $|f(t)| \leq Me^{st}$.

We define the function of complex variable $F(p) = \int_0^\infty f(t) \cdot e^{-pt}\, dt$ and we call it the Laplace transformation of the function $f(t)$.

- We note it $\mathcal{L}\{f(t)\} = F(p)$
- The function $f(t)$, which satisfied the conditions 1-3 is called the subject of LT
- The function $F(p)$ we called the image of the function $f(t)$ in LT
- LT is linear: $\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}, a, b \in C$

The basic vocabulary and grammar of the LT

0	$f(t)$	$\mathcal{L}\{f(t)\} = F(p) = \int_0^\infty f(t) \cdot e^{-pt}\, dt$
1.	c	$\frac{c}{p}$
2.	$t^n, n \in N$	$\frac{n!}{p^{n+1}}$
3.	e^{at}	$\frac{1}{p-a}$
4.	$t^n e^{at}, n \in N$	$\frac{n!}{(p-a)^{n+1}}$
5.	$\cos \omega t$	$\frac{p}{p^2 + \omega^2}$
6.	$\sin \omega t$	$\frac{\omega}{p^2 + \omega^2}$
7.	$e^{at} \cos \omega t$	$\frac{p-a}{(p-a)^2 + \omega^2}$
8.	$e^{at} \sin \omega t$	$\frac{\omega}{(p-a)^2 + \omega^2}$
9.	$f'(t)$	$pF(p) - f(0)$
10.	$f''(t)$	$p^2F(p) - pf(0) - f'(0)$
11.	$f'''(t)$	$p^3F(p) - p^2f(0) - pf'(0) - f''(0)$
12.	$f^{(n)}(t)$	$p^nF(p) - p^{n-1}f(0) - p^{n-2}f'(0) - \cdots - f^{(n-1)}(0)$
13.	$\int_0^t f(u)du$	$\frac{F(p)}{p}$
14.	$f(t-a); a \geq 0$	$e^{-ap}F(p)$
15.	$t \cdot f(t)$	$-F'(p)$

The reverse Laplace Transform

- The transition from the operator function $F(p)$ to the subject $f(t)$
- We note: $\mathcal{L}^{-1}\{F(p)\}$

The decomposition theorem

Let the operator function $F(p)$ has shape of the purely rational refracted function $F(p) = \frac{M(p)}{N(p)}$, where $M(p)$ and $N(p)$ are the polynomials, the order of the polynomial $M(p)$ is less then order of $N(p)$. We note p_k the poles of the function $F(p) = \frac{M(p)}{N(p)}$ then:

$$f(t) = \mathcal{L}^{-1}\{F(p)\} = \sum_{p_k} \operatorname*{res}_{p=p_k} [F(p)e^{pt}], t > 0.$$

If the function $F(p) = \frac{M(p)}{N(p)}$ with the real coefficients has a complex poles $p_{1,2} = \alpha \pm \beta i$, its enough to compute the residuum only for one root:

$$\operatorname*{res}_{p=\alpha+\beta i}[F(p)e^{pt}] + \operatorname*{res}_{p=\alpha-\beta i}[F(p)e^{pt}] = 2Re\left(\operatorname*{res}_{p=\alpha+\beta i}[F(p)e^{pt}]\right)$$

The solving of the Differential Equations using the Laplace Transform

Conversion the DE to an Algebraic

Space of subject	DE + Initial Conditions	The solving DE which satisfies IC
	$\mathcal{L}$ Laplace Transform of the DE	$\mathcal{L}^{-1}$ Laplace Transform of the solving
Space of images	Algebraic equation	Solving