EVROPSKÁ UNIE
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MINISTERSTVO ŠKOLSTVÍ, MLÁDEŽE A TĚLOVÝCHOVY

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Complex numbers (CN), basic concepts

- CN were not invented, first discussion about using in 16-th century
- Concept of CN was developed during more centuries $N \subset Z \subset Q \subset R \subset C$
- 18th century – big transition in research and using of CN – Carl Friedrich Gauss (established the term of *imaginary*)

Imaginary unit (Gauss, Cauchy)

$$x^2 + 1 = 0 \rightarrow x^2 = -1 \rightarrow |x| = \sqrt{-1} = i \rightarrow i^2 = -1$$

Note.:

- Electrical engineers prefer to reserve the symbol j, because i is symbol for current in electric circuits
- For us are both symbols allowable
- Matlab do not distinguish i or *j*

The rectangular form of CN

$$a = [a_1; a_2] = [a_1; 0] + [0; a_2] = a_1 + a_2 i = a_1 + a_2 j$$

The Gaussian complex plane of CN

A set of CN we can uniquely project to a set of plane points

$$a = [a_1; a_2]$$

We also define:

There are two CN $a = [a_1; a_2]$ a $b = [b_1; b_2]$

- The equivalence of two CN a,b: $a = b \Leftrightarrow a_1 = b_1 \wedge a_2 = b_2$
- The sum and substraction of two CN a,b: $a \pm b = [a_1 \pm b_1; a_2 \pm b_2]$
- The multiplication of two CN a,b: $a \cdot b = [a_1 b_1 - a_2 b_2; a_1 b_2 + a_2 b_1]$
- The division of two CN a,b: $\frac{a}{b} = \left[\frac{a_1 b_1 + a_2 b_2}{b_1^2 + b_2^2}; \frac{a_2 b_1 - a_1 b_2}{b_1^2 + b_2^2}\right]$
- The complex conjugate CN of CN a: $\bar{a} = [a_1; -a_2]$
- The reverse CN of CN a: $-a = [-a_1; -a_2]$
- The real number is the CN $[a_1; 0]$
- The pure imaginary CN is CN $[0; a_2]$
- The imaginary unit is a purely complex number $i = j = [0; 1]$

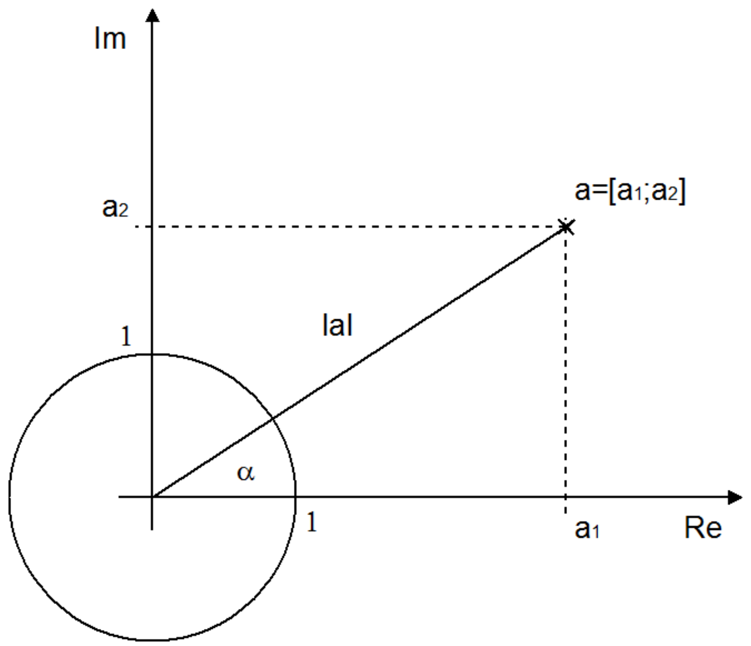
Definition:

Each arranged pair $[a_1; a_2]$ of real numbers a_1, a_2 is called a *complex number*

- We write $a = [a_1; a_2]$
- a_1 is the *real part of the comlex numberje* a, we note down $Re(a) = a_1$
- a_2 is the imaginary part of the complex number a, we note down $Im(a) = a_2$

The magnitude (The absolute value) of CN $a = [a_1; a_2]$ is called the real nonnegative number $|a| = \sqrt{a_1^2 + a_2^2}$, which expresses the distance of the CN from the origin of the coordinate system in the Gaussian plane

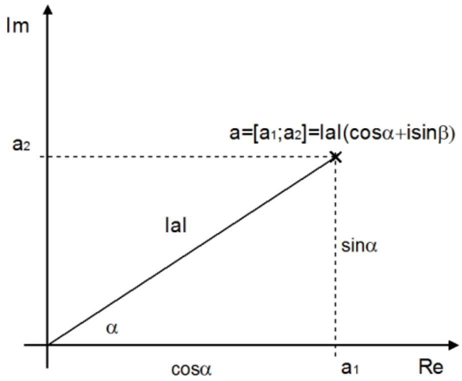
- The complex unit is every CN whose absolute value is equal to one
- The images of all complex units lies on a unit circle with the center at the origin of the coordinate system in the Gaussian plane



The goniometric form of the CN

$a = [a_1; a_2] = a_1 + a_2 i = |a|(\cos\alpha + i\sin\alpha)$, where α we called the direction (the argument) of the CN, which is define as: $\cos\alpha = \frac{a_1}{|a|} \wedge \sin\alpha = \frac{a_2}{|a|}$

The basic direction $\alpha \in \langle 0; 2\pi)$



The powers and roots of the CN

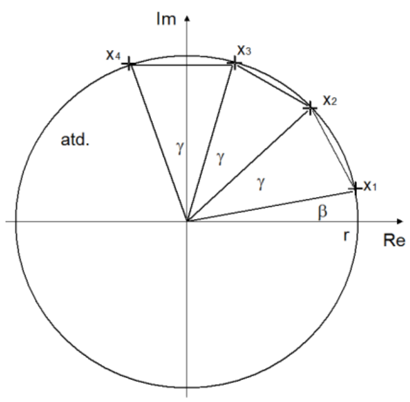
For any two CN

$$a = [a_1; a_2] = a_1 + a_2 i = |a|(\cos\alpha + i\sin\alpha)$$

$$b = [b_1; b_2] = b_1 + b_2 i = |b|(\cos\beta + i\sin\beta)$$

and for each $n \in N$ we apply:

- $a^n = |a|^n(\cos n\alpha + i\sin n\alpha)$ (Moivre theorem)
- $a \cdot b = |a||b|[\cos(\alpha+\beta) + i\sin(\alpha+\beta)]$
- $\frac{a}{b} = \frac{|a|}{|b|}[\cos(\alpha-\beta) + i\sin(\alpha-\beta)]; b \neq 0$
- $\sqrt[n]{a}$ is each from n roots of the binomial equation $x^n = a, a \in C, n \in N$, valid:
$$x = \sqrt[n]{|a|}\left[\cos\left(\frac{\alpha + 2k\pi}{n}\right) + i\sin\left(\frac{\alpha + 2k\pi}{n}\right)\right]$$
- In the Gaussian plane od CN are the projects of all n roots of the binomial equation are the peaks of the regular n-angle, which is inscribed in the circle with the center $[0; 0]$ and the radius is $\sqrt[n]{|a|}$, the complex function of $\sqrt[n]{a}$ belong into a group of so-called complex ambiguous functions, $\beta = \frac{\alpha}{n}, \gamma = \frac{2\pi}{n}, r = \sqrt[n]{|a|}$

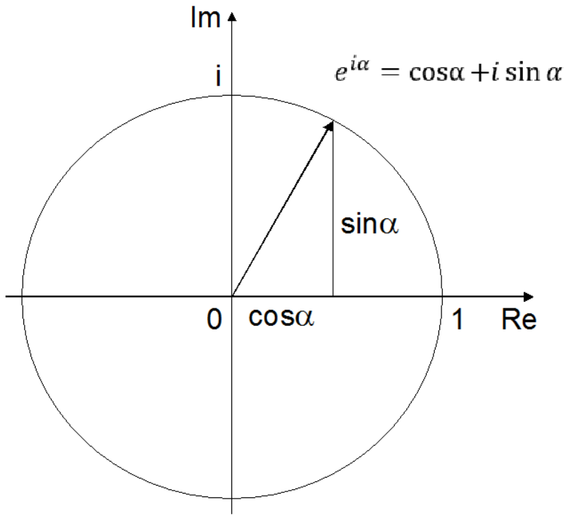


The polar (Euler) form of the CN

$$a = [a_1; a_2] = a_1 + a_2 i = |a|(\cos\alpha + i\sin\alpha) = |a|e^{i\alpha}$$

Where $|a|$ is magnitude of the CN a and α is its direction.

- The equation $e^{i\alpha} = (\cos\alpha + i\sin\alpha)$ is called the Euler formula and it determine the relationship between the geometric and the power functions



The functions of complex variable

The function $w = f(z)$, which is define on an area $\Omega \in C$ whose domain and range are subsets of the complex values is called the complex variables function.

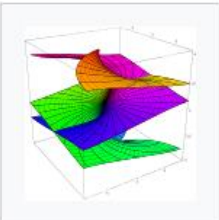
- The unique-valued complex function → if there is for each $z \in \Omega$ assigned just one complex value w (e.g.: $w = z + 1$)
- The multiple-valued complex function → if there is for each $z \in \Omega$ assigned more than one complex value (e.g.: $w = \sqrt{z+1}$)
- Algebraic shape of a complex function: $w = f(z) = u(x; y) + iv(x; y)$, where $z = x + iy$, $u(x; y), v(x; y)$ are the real functions of two variables x a y
- $u(x; y)$ is real part of $f(z)$, $Re(f) = u(x; y)$
- $v(x; y)$ is imaginary part of $f(z)$, $Im(f) = v(x; y)$

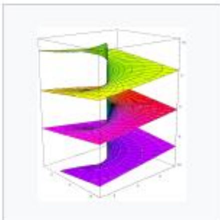
The list of the selected complex functions

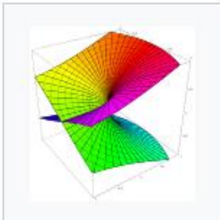
Title	Definition formula	Conditions
Exponential function	$e^z = e^x(\cos y + i \sin y)$	$z = x + iy; z \in C$
Cosines	$\cos z = \dfrac{e^{iz} + e^{-iz}}{2}$	$z = x + iy; z \in C$
Sinus	$\sin z = \dfrac{e^{iz} - e^{-iz}}{2i}$	$z = x + iy; z \in C$
Tangent	$\mathrm{tg}\ z = \dfrac{\sin z}{\cos z}$	$z \neq (2k+1)\dfrac{\pi}{2}; k \in Z$
Cotangent	$\mathrm{cotg}\ z = \dfrac{\cos z}{\sin z}$	$z \neq k\pi; k \in Z$
Cosines hyperbolical	$\cosh z = \dfrac{e^z + e^{-z}}{2}$	$z \in C$
Sinus hyperbolical	$\sinh z = \dfrac{e^z - e^{-z}}{2}$	$z \in C$
Logarithm function	$Ln\, z = ln\lvert z\rvert + iArg(z)$	$z \in C; z \neq 0$
Power function	$z^\alpha = e^{\alpha Ln\, z}$	$z, \alpha \in C; z \neq 0$

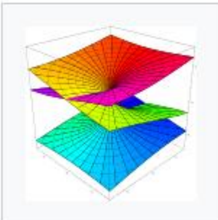
- Funkctions $e^z, \sinh z, \cosh z$ are the periodical with period $2\pi i$
- Funkctions $\sin z, \cos z$ are the periodical with period 2π
- Funkctions $Ln\, z, z^\alpha$ are multiple-valued, if we reduced to $Arg(z) \in \langle 0; 2\pi)$ we get the main branch of the function

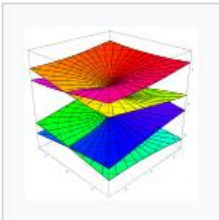
Note: the examples of the graphs of the selected complex functions


f(z) = arcsin z


f(z) = log z


f(z) = $z^{1/2}$


f(z) = $z^{1/3}$


f(z) = $z^{1/4}$

The limit of the function of the complex variable

- Concepts: $|z - a| < \varepsilon \rightarrow$ the epsilon-surrounding of the point a, $0 < |z - a| < \varepsilon \rightarrow$ the ring-surrounding of the point a
- Let $z = x + iy$ a $f(z) = u(x;y) + iv(x;y)$.

Funkction of the complex variable has in a point $a = a_1 + ia_2$ limit just when have the real functions $u(x;y)$ a $v(x;y)$ limit in the point $[a_1; a_2]$.

- The equation is true: $\lim\limits_{z\to a} f(z) = \lim\limits_{[x;y]\to[a_1;a_2]} u(x;y) + \mathrm{i} \cdot \lim\limits_{[x;y]\to[a_1;a_2]} v(x;y)$

Continuity of the function of a complex variable

- The function $f(z)$ of a complex variable is called continuous at a complex point a, if it has a limit at this point and if: $\lim_{z \to a} f(z) = f(a)$.
- The function $f(z)$ of a complex variable is called continuous on the set $M \subset C$, if it is continuous at each point of the set M.

The derivative of the complex variable functions

- Same implementation as with real functions
- The derivative of the complex function $f(z)$ at the complex point a we call the number $f'(a)$ which is given by the relation $f'(a) = \lim_{z \to a} \frac{f(z) - f(a)}{z - a}$
- For the calculations of the derivations, there are the same formulas apply to real variable functions (see MI and M2)
- If $f'(a)$ exists, we say the function $f(z)$ is the differentiable in the point a
- If $f'(z)$ exists in the point a and some of its surrounding, the function $f(z)$ is called Holomorphic at point a

The relation between the derivation of the function of the complex variable and the partial derivatives of its real and imaginary part is solved using the Cauchy-Riemann's conditions:

$\frac{\partial u}{\partial x}(A) = \frac{\partial v}{\partial y}(A) \ \wedge \frac{\partial u}{\partial y}(A) = -\frac{\partial v}{\partial x}(A).$

- Thus $f'(a) = \frac{\partial u}{\partial x}(A) + i\frac{\partial v}{\partial x}(A) = \frac{\partial u}{\partial y}(A) - i\frac{\partial v}{\partial y}(A)$

The integral of the function of the complex variable – calculation using curve parameterization

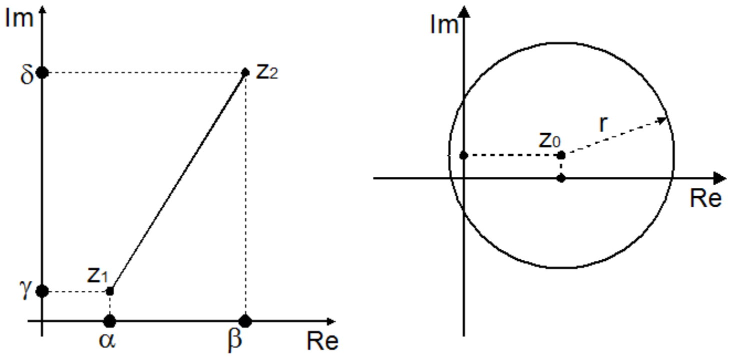
- When we integrate the functions that are not holomorphic to C $\{Re(z); Im(z); |z|; \ \bar{z}\}$

Let $\Gamma: R \to C$ is the complex function of the real variable t. $\Gamma: z(t) = x(t) + iy(t); \ t \in \langle \alpha; \beta \rangle$,

where $x(t), y(t)$ are continuous real functions of one variable t that their derivations $x'(t), y'(t)$ are in parts continuous. Then we can say that Γ is in parts smooth oriented curve in complex plane started in the point $z(\alpha)$ and ended in the point $z(\beta)$.

We will consider only the segments of the line and the circles (see the picture of parametrization)

- The line segment with the limit points z_1, z_2 has the parametric equation $z(t) = z_1 + (z_2 - z_1)t; t \in \langle 0; 1 \rangle$
- The sections laying on the axis:
  - On x-axis between points $z_1 = \alpha; \ z_2 = \beta$: $z(t) = t; t \in \langle \alpha; \beta \rangle$
  - On y-axis between points $z_1 = i\gamma; \ z_2 = i\delta$: $z(t) = it; t \in \langle \gamma; \delta \rangle$
- A positively oriented circle with a center at point z_0 of radius r has a parametric equation $z(t) = z_0 + re^{it}; \ t \in \langle 0; 2\pi \rangle$



Parametrization

Let $w = f(z)$ is the unique-valued complex function and continuous on the curve Γ. After it the integral from the function f along the curve Γ is defined:

$$\oint_{\Gamma} f(z)dz = \int_{\alpha}^{\beta} f[z(t)] \cdot z'(t)dt$$

To the integral of the function of the complex variable we can apply the analogy properties of the integral of the real functions:

- $\oint_{\Gamma} [kf_1(z) + lf_2(z)]dz = k\oint_{\Gamma} f_1(z)dz + l\oint_{\Gamma} f_2(z)\,dz$
- If curve Γ consists from the curves Γ_1 a Γ_2, which have the unique common point (for n curves analogically)
  - $\oint_{\Gamma} f(z)dz = \oint_{\Gamma_1} f(z)dz + \oint_{\Gamma_2} f(z)dz$
- If the curve Γ_1 is negative oriented curve to the curve Γ
  - $\oint_{\Gamma} f(z)dz = -\oint_{\Gamma_1} f(z)dz$

The integration of the functions of the complex variables – the calculation without the curve parametrization

- When we integrate the holomorphic functions at C, we do not need to use the curve parametrization
- We use the Cauchy teorem or the Cauchy formula or the Residual theorem (see later)

Cauchy theorem:

If the function $f(z)$ is holomorphic in the simply smooth region Ω, where the curve Γ lies, then the integral value $\oint_{\Gamma} f(z)dz$ does not depend on the shape of the curve Γ, only at its limit points then:

$$\oint_{\Gamma} f(z)dz = F(z_2) - F(z_1),$$

where z_1 is initial point and z_2 is terminal point of the curve Γ.

- For function $F(z)$ holds $F'(z) = f(z)$.
- For closed curve Γ in this area holds $\oint_{\Gamma} f(z)dz = 0$.

Cauchy formula:

If the function $f(z)$ is holomorphic in the simply smooth region Ω, where the curve Γ lies, then:

$$\oint_{\Gamma} \frac{f(z)}{z - z_0} dz = \begin{cases} 2\pi i f(z_0); & if\ z_0\ is\ in\ \Gamma \\ 0; & if\ z_0\ is\ out\ of \Gamma \end{cases}$$

Residuum theorem (see lately)