Laplaceova Transformace

Laplaceova Transformace (LT)

Nechť funkce $f: R \to R$ splňuje následující podmínky:

1. Funkce $f(t)$ je alespoň po částech spojitá,
2. $f(t) = 0$ pro $t < 0$,
3. Funkce $f(t)$ je exponenciálního řádu, tzn. Existují konstanty $M, s, t_0 \in R, t_0 > 0$, takové, že pro všechna $t \geq t_0$ platí $|f(t)| \leq Me^{st}$.

Potom definujeme funkci komplexní proměnné $F(p) = \int_0^\infty f(t) \cdot e^{-pt}\, dt$ a říkáme jí Laplaceova transformace funkce $f(t)$.

- Značíme $\mathcal{L}\{f(t)\} = F(p)$
- Funkce $f(t)$, která splňuje podmínky 1-3 se nazývá předmět
- Funkce $F(p)$ se nazývá obraz funkce $f(t)$ v LT
- LT je lineární, platí:

$$\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}, a, b \in C$$

PK1

Základní slovník a gramatika LT

0	$f(t)$	$\mathcal{L}\{f(t)\} = F(p) = \int_0^\infty f(t) \cdot e^{-pt}\, dt$
1.	c	$\frac{c}{p}$
2.	$t^n, n \in N$	$\frac{n!}{p^{n+1}}$
3.	e^{at}	$\frac{1}{p-a}$
4.	$t^n e^{at}, n \in N$	$\frac{n!}{(p-a)^{n+1}}$
5.	$\cos \omega t$	$\frac{p}{p^2+\omega^2}$
6.	$\sin \omega t$	$\frac{\omega}{p^2+\omega^2}$
7.	$e^{at} \cos \omega t$	$\frac{p-a}{(p-a)^2+\omega^2}$
8.	$f'(t)$	$pF(p) - f(0)$
9.	$f''(t)$	$p^2F(p) - pf(0) - f'(0)$
10.	$f'''(t)$	$p^3F(p) - p^2f(0) - pf'(0) - f''(0)$
11.	$f^{(n)}(t)$	$p^nF(p) - p^{n-1}f(0) - p^{n-2}f'(0) - \cdots - f^{(n-1)}(0)$
12.	$\int_0^t f(u)du$	$\frac{F(p)}{p}$
13.	$f(t-a); a \geq 0$	$e^{-ap}F(p)$
14.	$t \cdot f(t)$	$-F'(p)$

PK2

Zpětná Laplaceova Transformace

- Přechod od operátorové funkce $F(p)$ ke vzoru $f(t)$
- Značíme: $\mathcal{L}^{-1}\{F(p)\}$

Věta o rozkladu

Nechť operátorové funkce $F(p)$ má tvar ryze racionální lomené funkce $F(p) = \frac{M(p)}{N(p)}$, kde $M(p)$ a $N(p)$ jsou polynomy, stupeň polynomu $M(p)$ je menší než stupeň $N(p)$. Označíme p_k póly funkce $F(p) = \frac{M(p)}{N(p)}$, pak platí:

$$f(t) = \mathcal{L}^{-1}\{F(p)\} = \sum_{p_k} \operatorname*{res}_{p=p_k} [F(p)e^{pt}], t > 0.$$

Jestli-že funkce $F(p) = \frac{M(p)}{N(p)}$ s reálnými koeficienty má komplexní póly $p_{1,2} = \alpha \pm \beta i$, stačí vypočítat reziduum pouze pro jeden kořen:

$$\operatorname*{res}_{p=\alpha+\beta i}[F(p)e^{pt}] + \operatorname*{res}_{p=\alpha-\beta i}[F(p)e^{pt}] = 2Re\left(\operatorname*{res}_{p=\alpha+\beta i}[F(p)e^{pt}]\right)$$

Př3 Př4 Př5 Př6

Řešení Diferenciálních rovnic Laplaceovou Transformací

Převod DR na Algebraické

Prostor Předmětu	D Rce + Počáteční Podmínky ⬇	Řešení DR vyhovující PP ⬆
	$\mathcal{L}$ Laplaceova Transformace rovnice ⬇	$\mathcal{L}^{-1}$ Laplaceova Transformace řešení
Prostor Obrazů	Algebraická rovnice ➡	⬆ Řešení

Př7 Př8 Př9

PŘ1 PŘÍMÝM výpočtem určete obraz fce $f(t)=t$

fce $f(t)=\begin{cases} t; & t\ge 0 \\ 0; & t<0 \end{cases}$ → fce je spojitá, derivace po částech spojitá ($f'(t)=\begin{cases} 1; & t\ge 0 \\ 0; & t<0 \end{cases}$)

→ splňuje 2

→ splňuje 3 $\quad t<e^{t}$ pro $t\ge 0$

$$\mathcal{L}\{t\}=\int_0^\infty t\cdot e^{-pt}\,dt=\lim_{A\to\infty}\int_0^A t\,e^{-pt}\,dt=\left|\begin{matrix} u=t & u'=1 \\ v'=e^{-pt} & v=-\frac{1}{p}e^{-pt}\end{matrix}\right|=$$

$$=\lim_{A\to\infty}\left\{\left[-\frac{t}{p}e^{-pt}\right]_0^A-\int_0^A\left(-\frac{1}{p}\right)\cdot e^{-pt}\,dt\right\}=\lim_{A\to\infty}\left\{\left[-\frac{A}{p}e^{-pA}\right]-[0]+\frac{1}{p}\int_0^A e^{-pt}\,dt\right\}=$$

$$=\lim_{A\to\infty}\left\{-\frac{A}{p\cdot e^{pA}}+\frac{1}{p}\left[\left(-\frac{1}{p}\right)e^{-pt}\right]_0^A\right\}=\left|\begin{matrix}\text{pro } \operatorname{Re} p>0 \\ \text{je } \lim\limits_{A\to\infty}\frac{A}{pe^{pA}}=0\end{matrix}\right|=$$

$$=\lim_{A\to\infty}\left(-\frac{1}{p^2}\left[\underbrace{e^{-pA}}_{0}-\underbrace{e^{0}}_{1}\right]\right)=-\frac{1}{p^2}(0-1)=\underline{\underline{\frac{1}{p^2}}}$$

PŘ2 pomocí tabulky určete obraz fce $f(t)=t^3+3t^2+2t+1$

dle vz 2.

$$\mathcal{L}\{t^3+3t^2+2t+1\}=\overset{\frac{3!}{p^4}=\frac{6}{p^4}}{\mathcal{L}\{t^3\}}+3\overset{\frac{2!}{p^3}=\frac{2}{p^3}}{\mathcal{L}\{t^2\}}+2\overset{\frac{1}{p^2}}{\mathcal{L}\{t\}}+\overset{\frac{1}{p}}{\mathcal{L}\{1\}}=$$

$$=\frac{3!}{p^4}+3\cdot\frac{2!}{p^3}+2\cdot\frac{1}{p^2}+\frac{1}{p}=\underline{\underline{\frac{6}{p^4}+\frac{6}{p^3}+\frac{2}{p^2}+\frac{1}{p}}}$$

PŘ3 Nalezněte vzor fce $F(p)=\dfrac{1}{p^2(p-4)}$

1. rozklad na parciální zlomky $\dfrac{1}{p^2(p-4)}=\dfrac{A}{p^2}+\dfrac{B}{p}+\dfrac{C}{p-4}=\dfrac{A(p-4)+Bp(p-4)+Cp^2}{p^2(p-4)}=$

$$=\frac{Ap-4A+Bp^2-4Bp+Cp^2}{p^2(p-4)}$$

p^2: $0=B+C$ $\quad C=-B=\frac{1}{16}$

p: $0=A-4B$ $\quad 0=-\frac{1}{4}-4B\Rightarrow B=-\frac{1}{16}$

p^0: $\underline{1=-4A}$ $\quad\Rightarrow A=-\frac{1}{4}$

$$\mathcal{L}^{-1}\left\{\frac{1}{p^2(p-4)}\right\}=\mathcal{L}^{-1}\left\{-\frac{1}{4p^2}-\frac{1}{16p}+\frac{1}{16(p-4)}\right\}=-\frac{1}{4}\mathcal{L}^{-1}\left\{\frac{1}{p^2}\right\}-\frac{1}{16}\mathcal{L}^{-1}\left\{\frac{1}{p}\right\}+\frac{1}{16}\mathcal{L}^{-1}\left\{\frac{1}{p-4}\right\}=$$

$$=\underline{\underline{-\frac{1}{4}t-\frac{1}{16}\cdot 1+\frac{1}{16}\cdot e^{4t}}}$$

2. Pomocí vět o rozkladu póly $p^2(p-4)=0$ $p_{1,2}=0$; $p_3=4$
2.ř 1.ř

$$\mathcal{L}^{-1}\left\{\frac{1}{p^2(p-4)}\right\} = \operatorname*{res}_{p=0} F(p)e^{pt} + \operatorname*{res}_{p=4} F(p)e^{pt} = \lim_{p\to 0}\left(p^2\frac{e^{pt}}{p^2(p-4)}\right)' +$$

$$+\lim_{p\to 4}(p-4)\frac{e^{pt}}{p^2(p-4)} = \lim_{p\to 0}\frac{e^{pt}\cdot t(p-4)-e^{pt}}{(p-4)^2} + \lim_{p\to 4}\frac{e^{pt}}{p^2} =$$

$$= \frac{1t(-4)-1}{16} + \frac{e^{4t}}{16} = -\frac{1}{4}t - \frac{1}{16} + \frac{1}{16}e^{4t}$$

Př 4) Pomocí tabulky vypočtěte vzory těchto obrazů $F(p)$

a) $F(p)=\frac{1}{p^2+4}$ $\mathcal{L}^{-1}\{F(p)\} = \mathcal{L}^{-1}\left\{\frac{1}{p^2+4}\right\} = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{p^2+4}\right\} = \frac{1}{2}\sin 2t$

b) $F(p)=\frac{p+1}{p^2+2p+2}$ $\mathcal{L}^{-1}\{F(p)\} = \mathcal{L}^{-1}\left\{\frac{p+1}{p^2+2p+2}\right\} = \mathcal{L}^{-1}\left\{\frac{p+1}{(p+1)^2+1}\right\} = e^{-t}\cos t$

c) $F(p)=\frac{5p+3}{(p-1)(p^2+2p+5)}$ $\frac{A}{p-1} + \frac{Bp+C}{p^2+2p+5} = \frac{Ap^2+2Ap+5A+Bp^2-Bp+Cp-C}{(p-1)(p^2+2p+5)}$

p^2: $0 = A + B$
p: $5 = 2A - B + C$
p^0: $3 = 5A \quad - C$ $C = 5A-3$
$5 = 2A + A + 5A - 3$
$8 = 8A$ $A=1$ $B=-1$ $C=2$

$$\mathcal{L}^{-1}\left\{\frac{5p+3}{(p-1)(p^2+2p+5)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{p-1} + \frac{-p+2}{p^2+2p+5}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{p-1}\right\} + \mathcal{L}^{-1}\left\{\frac{-p+2}{(p+1)^2+4}\right\} =$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{p-1}\right\} - \mathcal{L}^{-1}\left\{\frac{p+1}{(p+1)^2+4}\right\} + \frac{3}{2}\mathcal{L}^{-1}\left\{\frac{2}{(p+1)^2+4}\right\} = e^{t} - e^{-t}\cos 2t + \frac{3}{2}e^{-t}\sin 2t$$

(PŘ 5) Pomocí věty o rozkladu nalezněte vzor daného obrazu $F(p)$

$$F(p) = \frac{5p+3}{(p-1)(p^2+2p+5)} \qquad (p-1)(p^2+2p+5)=0$$

$$\Leftrightarrow p_1 = 1 \vee p^2+2p+5=0$$
$1R$

$$p_{2,3} = \frac{-2 \pm \sqrt{-16}}{2} \quad \begin{cases} p_2 = -1+2i & 1R \\ p_3 = -1-2i & 1R \end{cases}$$

$$\operatorname*{res}_{p=p_2} F(p)e^{pt} = \lim_{p\to -1+2i}\left[\cancel{(p+1-2i)}\frac{(5p+3)e^{pt}}{(p-1)\cancel{(p+1-2i)}(p+1+2i)}\right] = \frac{(-5+10i+3)e^{(-1+2i)t}}{(-2+2i)\cdot 4i} =$$

$$= \frac{-1+5i}{-4-4i}e^{(-1+2i)t} = \frac{(-1+5i)(-4+4i)}{(-4-4i)(-4+4i)}e^{(-1+2i)t} = \frac{4-4i-20i+\overset{-20}{20i^2}}{\underset{+16}{16-16i^2}}e^{(-1+2i)t} =$$

$$= \frac{-16-24i}{32}e^{(-1+2i)t} = -\frac{2+3i}{4}e^{(-1+2i)t}$$

platí, že $\displaystyle\operatorname*{res}_{p=p_2} F(p)e^{pt} + \operatorname*{res}_{p=p_3} F(p)e^{pt} = 2\operatorname{Re}\operatorname*{res}_{p=p_2} F(p)e^{pt} =$

$$= 2\operatorname{Re}\left[-\frac{2+3i}{4}e^{-t}\cdot e^{2it}\right] = 2\operatorname{Re}\left[\left(-\frac{1}{2}-\frac{3}{4}i\right)e^{-t}\cdot(\cos 2t + i\sin 2t)\right] =$$

$$= 2\operatorname{Re}\left[-\frac{1}{2}\cos 2t\, e^{-t} - \cancel{\frac{1}{2}i\sin 2t\cdot e^{-t}} - \cancel{\frac{3}{4}i\cos 2t\, e^{-t}} + \frac{3}{4}\sin 2t\cdot e^{-t}\right] =$$

$$= 2\left(-\frac{1}{2}\cos 2t\, e^{-t} + \frac{3}{4}\sin 2t\, e^{-t}\right) = -e^{-t}\cos 2t + \frac{3}{2}e^{-t}\sin 2t$$

$$\operatorname*{res}_{p=p_1} F(p)e^{pt} = \lim_{p\to 1}\left[\cancel{(p-1)}\frac{5p+3}{\cancel{(p-1)}(p^2+2p+5)}e^{pt}\right] = \frac{8}{8}e^{t} = e^{t}$$

$$\mathcal{L}^{-1}\left\{\frac{5p+3}{(p-1)(p^2+2p+5)}\right\} = \underline{e^{t} - e^{-t}\cos 2t + \frac{3}{2}e^{-t}\sin 2t}$$

PŘ6 Laplaceova transformace derivací

① $\mathcal{L}\{f'(t)\} = \int_0^\infty e^{-pt} \cdot f'(t)\,dt = \left[e^{-pt} \cdot f(t)\right]_0^\infty + p\underbrace{\int_0^\infty e^{-pt} f(t)\,dt}_{\mathcal{L}\{f(t)\}} = 0 - f(0) + p\mathcal{L}\{f(t)\} \ominus$

$u = e^{-pt} \quad u' = -pe^{-pt}$

$v' = f'(t) \quad v = f(t)$

$\ominus - f(0) + p\underbrace{\mathcal{L}\{f(t)\}}_{F(p)} = p \cdot F(p) - f(0)$

② $\mathcal{L}\{f''(t)\} = \int_0^\infty e^{-pt} f''(t)\,dt = \left[e^{-pt} \cdot f'(t)\right]_0^\infty + p\underbrace{\int_0^\infty e^{-pt} \cdot f'(t)\,dt}_{\mathcal{L}\{f'(t)\}} = 0 - f'(0) + p\mathcal{L}\{f'(t)\} \ominus$

$u = e^{-pt} \quad u' = -pe^{-pt}$

$v' = f''(t) \quad v = f'(t)$

$\ominus - f'(0) + p\left[p\mathcal{L}\{f(t)\} - f(0)\right] = -f'(0) - pf(0) + p^2\underbrace{\mathcal{L}\{f(t)\}}_{F(p)} =$

$= p^2 F(p) - pf(0) - f'(0)$

$\Rightarrow$ ③ $\mathcal{L}\{f'''(t)\} = p^3 F(p) - p^2 f(0) - pf'(0) - f''(0)$

$\Rightarrow$ ⓝ $\mathcal{L}\{f^{(n)}(t)\} = p^n F(p) - p^{n-1} f(0) - p^{n-2} f'(0) - \ldots - f^{(n-1)}(0)$

PŘ7 LT řeší DR

7a) $y' - 2y = 1 \quad ; y(0) = -2$

$\mathcal{L}\{y\} = Y(p) \; ; \mathcal{L}\{y'\} = pY(p) - (-2) \; ; \mathcal{L}\{1\} = \frac{1}{p}$

$\Rightarrow pY(p) + 2 - 2Y(p) = \frac{1}{p} \Rightarrow pY(p) - 2Y(p) = \frac{1}{p} - 2$

$Y(p)(p-2) = \frac{1}{p} - 2$

$Y(p) = \frac{\frac{1}{p} - 2}{p-2} = \frac{\frac{1-2p}{p}}{p-2} = \frac{1-2p}{p(p-2)}$

$$Y(p)=\frac{1-2p}{p(p-2)}=\frac{A}{p}+\frac{B}{p-2}=\frac{Ap-2A+Bp}{p(p-2)}$$

$p: -2=A+B \quad B=-2+\frac{1}{2}=-\frac{3}{2}$
$p^0: 1=-2A \Rightarrow A=-\frac{1}{2}$

$$Y(p)=-\frac{1}{2p}-\frac{3}{2(p-2)}$$

$$y(t)=\mathcal{L}^{-1}\left\{-\frac{1}{2p}-\frac{3}{2(p-2)}\right\}=-\frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{p}\right\}-\frac{3}{2}\mathcal{L}^{-1}\left\{\frac{1}{p-2}\right\}=-\frac{1}{2}\cdot 1-\frac{3}{2}e^{2t}$$

Rj: $\underline{y(t)=-\frac{1}{2}-\frac{3}{2}e^{2t}}$

7 b) $y'''+y'=e^{2t}$; $y(0)=y'(0)=y''(0)=0$

$$\mathcal{L}\{y'''(t)\}=p^3Y(p)-p^2y(0)-py'(0)-y''(0)=p^3Y(p)$$
$$\mathcal{L}\{y'(t)\}=pY(p)-y(0)=pY(p)$$
$$\mathcal{L}\{e^{2t}\}=\frac{1}{p-2}$$

$$p^3Y(p)+pY(p)=\frac{1}{p-2}$$
$$Y(p)(p^3+p)=\frac{1}{p-2}$$
$$Y(p)=\frac{1}{(p-2)(p^3+p)}=\frac{1}{(p-2)p(p^2+1)}$$

poly: $(p-2)p(p-i)(p+i)=0 \quad p_1=2 \; 1R \quad p_2=0 \; 1R \quad p_3=i \; 1R \quad p_4=-i \; 1R$

$$\operatorname*{res}_{p=2} F(p)e^{pt}=\lim_{p\to2}\left[\cancel{(p-2)}\frac{1}{\cancel{(p-2)}p\underbrace{(p-i)(p+i)}_{(p^2+1)}}e^{pt}\right]=\frac{1}{10}e^{2t}$$

$$\operatorname*{res}_{p=0} F(p)e^{pt}=\lim_{p\to0}\left[\cancel{p}\frac{1}{(p-2)\cancel{p}(p^2+1)}e^{pt}\right]=-\frac{1}{2}e^0=-\frac{1}{2}$$

$$\operatorname*{res}_{p=i} F(p)e^{pt}=\lim_{p\to i}\left[\cancel{(p-i)}\frac{1}{(p-2)p\cancel{(p-i)}(p+i)}e^{pt}\right]=\frac{1}{(i-2)\underbrace{2i^2}_{-2}}e^{it}=\frac{i+2}{(-2)(i-2)(i+2)}e^{it}=$$

$$=\frac{i+2}{10}(\cos t+i\sin t)=\left(\frac{1}{5}+\frac{1}{10}i\right)(\cos t+i\sin t)=\underbrace{\frac{1}{5}\cos t-\frac{1}{10}\sin t}_{Re}+i\left(\frac{1}{10}\cos t+\frac{1}{5}\sin t\right)$$

$$\operatorname*{res}_{p=i} F(p)e^{pt}+\operatorname*{res}_{p=-i} F(p)e^{pt}=2\,Re\operatorname*{res}_{p=i} F(p)e^{pt}=2\left(\frac{1}{5}\cos t-\frac{1}{10}\sin t\right)$$

$$y(t)=\mathcal{L}^{-1}\left\{\frac{1}{p(p-2)(p-i)(p+i)}\right\}=\underline{\frac{1}{10}e^{2t}-\frac{1}{2}+\frac{2}{5}\cos t-\frac{1}{5}\sin t}$$

7 c) $y'' + 4y' + 8y = 0 \qquad y(0) = 0 \, ; \, y'(0) = 1$

$\mathcal{L}\{y''(t)\} = p^2 Y(p) - p \cdot 0 - 1 \, ; \; \mathcal{L}\{y'(t)\} = p\,Y(p) - 0 \, ; \; \mathcal{L}\{y(t)\} = Y(p)$

$$p^2 Y(p) - 1 + 4p\,Y(p) + 8\,Y(p) = 0$$

$$Y(p)\left(p^2 + 4p + 8\right) = 1$$

$$Y(p) = \frac{1}{p^2+4p+8}$$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{p^2+4p+8}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(p+2)^2+4}\right\} = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{(p+2)^2+4}\right\} = \underline{\underline{\frac{1}{2}e^{-2t}\sin 2t}}$$

7 d) $y'' - 18y' + 72y = -36t\,e^{6t} \qquad ; \, y(0) = 0 \, ; \, y'(0) = 1$

$\mathcal{L}\{-36t\,e^{6t}\} = \dfrac{1}{(p-6)^2}$ ostatní stejné jako v c

$$p^2 Y(p) - 1 - 18p\,Y(p) + 72\,Y(p) = -36 \cdot \frac{1}{(p-6)^2}$$

$$Y(p)\left(p^2 - 18p + 72\right) = -\frac{36}{(p-6)^2} + 1 = \frac{-36 + p^2 - 12p + 36}{(p-6)^2} = \frac{p(p-12)}{(p-6)^2}$$

$$Y(p) = \frac{p(p-12)}{(p-6)^2(p^2-18p+72)} = \frac{p\cancel{(p-12)}}{(p-6)^2\cancel{(p-12)}(p-6)} = \frac{p}{(p-6)^3}$$

$y(t) = \mathcal{L}^{-1}\left\{\dfrac{p}{(p-6)^3}\right\} \qquad p = 6$ pól 3R

$$\operatorname*{res}_{p=6}\left[F(p)\cdot e^{pt}\right] = \frac{1}{2}\lim_{p\to 6}\left[\cancel{(p-6)^3}\frac{p}{\cancel{(p-6)^3}}e^{pt}\right]'' = \frac{1}{2}\lim_{p\to 6}\left[p \cdot e^{pt}\right]'' = \frac{1}{2}\lim_{p\to 6}\left(e^{pt} + pt\,e^{pt}\right)' =$$

$$= \frac{1}{2}\lim_{p\to 6}\left[te^{pt} + te^{pt} + pt^2e^{pt}\right] = \frac{1}{2}\lim_{p\to 6}\left[2te^{pt} + pt^2e^{pt}\right] = \frac{1}{2}\left(2te^{6t} + 6t^2e^{6t}\right)$$

$$\underline{\underline{y(t) = te^{6t} + 3t^2e^{6t}}}$$

7e) $y'' - 6y' + 9y = 0 \qquad y(0) = 1,\ y'(0) = 4$

$\mathcal{L}\{y''(t)\} = p^2 Y(p) - p - 4 \ ; \ \mathcal{L}\{y'(t)\} = pY(p) - 1 \ ; \ \mathcal{L}\{y(t)\} = Y(p)$

$p^2 Y(p) - p - 4 - 6[pY(p) - 1] + 9Y(p) = 0$

$\underline{p^2 Y(p)} - p - 4 - \underline{6pY(p)} + 6 + \underline{9Y(p)} = 0$

$Y(p)(p^2 - 6p + 9) = p - 2 \Rightarrow Y(p) = \dfrac{p-2}{p^2-6p+9} = \dfrac{p-2}{(p-3)^2}$ pól $p = 3$ 2.R

$y(t) = \mathcal{L}^{-1}\left\{\dfrac{p-2}{(p-3)^2}\right\}$

$\operatorname{res}_{p=3} F(p)e^{pt} = \lim_{p\to 3}\left[\cancel{(p-3)^2}\dfrac{p-2}{\cancel{(p-3)^2}}\cdot e^{pt}\right]' = \lim_{p\to 3}\left[(p-2)e^{pt}\right]' =$

$= \lim_{p\to 3}\left[e^{pt} + (p-2)t\cdot e^{pt}\right] = e^{3t} + t\cdot e^{3t}$

$\underline{\underline{y(t) = e^{3t} + te^{3t}}}$

7f) $y'' - 4y = 4t \qquad ; y(0) = 1 \ ; y'(0) = 0$

$\mathcal{L}\{y''(t)\} = p^2 Y(p) - p \ ; \ \mathcal{L}\{y(t)\} = Y(p) \ ; \mathcal{L}\{t\} = \dfrac{1}{p^2}$

$p^2 Y(p) - p - 4Y(p) = 4\cdot\dfrac{1}{p^2}$

$Y(p)(p^2 - 4) = \dfrac{4}{p^2} + p = \dfrac{4+p^3}{p^2} \Rightarrow Y(p) = \dfrac{4+p^3}{p^2(p-2)(p+2)}$ $\quad p=0$ 2R $\quad p=2$ 1R $\quad p=-2$ 1.R

$\operatorname{res}_{p=0}\left(F(p)e^{pt}\right) = \lim_{p\to 0}\left[p^2\dfrac{p^3+4}{p^2(p-2)(p+2)}e^{pt}\right]' = \lim_{p\to 0}\left(\dfrac{p^3+4}{p^2-4}e^{pt}\right)' = \lim_{p\to 0}\left[\dfrac{3p^2(p^2-4) - (p^3+4)\cdot 2p}{(p^2-4)^2}e^{pt} + \right.$

$\left. + \dfrac{p^3+4}{p^2-4}t\cdot e^{pt}\right] = -t\cdot e^0 = -t$

$\operatorname{res}_{p=2}\left(F(p)e^{pt}\right) = \lim_{p\to 2}\left[\cancel{(p-2)}\dfrac{p^3+4}{p^2\cancel{(p-2)}(p+2)}e^{pt}\right] = \dfrac{12}{16}e^{2t} = \dfrac{3}{4}e^{2t}$

$\operatorname{res}_{p=-2}\left(F(p)e^{pt}\right) = \lim_{p\to -2}\left[\cancel{(p+2)}\dfrac{p^3+4}{p^2(p-2)\cancel{(p+2)}}e^{pt}\right] = \dfrac{-4}{-16}e^{-2t} = \dfrac{1}{4}e^{-2t}$

$y(t) = \mathcal{L}^{-1}\left\{\dfrac{4+p^3}{p^2(p-2)(p+2)}\right\} = \underline{\underline{\dfrac{4}{3}e^{2t} + \dfrac{1}{4}e^{-2t} - t}}$

108 $y' + 3y = 13\sin 2t; \quad y(0) = 6$

$$\mathcal{L}\{y'(t)\} + 3\mathcal{L}\{y(t)\} = 13\mathcal{L}\{\sin 2t\}$$

$$sY(s) - \underset{6}{y(0)} + 3Y(s) = 13\frac{2}{s^2+4}$$

$$Y(s)(s+3) = \frac{26}{s^2+4} + 6 \Rightarrow Y(s) = \frac{26}{(s^2+4)(s+3)} + \frac{6}{s+3} = \frac{26+6s^2+24}{(s^2+4)(s+4)} = \frac{6s^2+50}{(s^2+4)(s+4)}$$

$$Y(s) = \frac{6s^2+50}{(s^2+4)(s+3)} = \frac{A}{s+3} + \frac{Bs+C}{s^2+4} = \frac{As^2+4A+Bs^2+3Bs+Cs+3C}{(s+3)(s^2+4)} = \frac{8}{s+3} + \frac{-2s+6}{s^2+4}$$

$$\begin{aligned} s^2 &: 6 = A + B \\ s &: 0 = 3B + C \\ s^0 &: 50 = 4A + 3C \end{aligned} \qquad \begin{aligned} -18 &= -3A - 3B \\ 0 &= 3B + C \\ \hline -18 &= -3A + C \quad / \cdot (-3) \\ 50 &= 4A + 3C \\ \hline 104 &= 13A \quad A = \frac{104}{13} = 8 \quad B = -2 \quad C = 6 \end{aligned}$$

$$y(t) = 8\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\} - 2\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} + 3\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\} = \underline{8e^{-3t} - 2\cos 2t + 3\sin 2t}$$

109 $y'' - 3y' + 2y = e^{-4t} \quad y(0) = 1; \ y'(0) = 5$

$$\mathcal{L}\{y''(t)\} - 3\mathcal{L}\{y(t)\} + 2\mathcal{L}\{y\} = \mathcal{L}\{e^{-4t}\}$$

$$s^2Y(s) - s\underset{1}{y(0)} - \underset{5}{y'(0)} - 3[sY(s) - \underset{1}{y(0)}] + 2Y(s) = \frac{1}{s+4}$$

$$Y(s)[s^2 - 3s + 2] = \frac{1}{s+4} + s + 2$$

$$Y(s) = \frac{1}{(s+4)(s^2-3s+2)} + \frac{s+2}{s^2-3s+2} = \frac{1+\overbrace{(s+2)(s+4)}^{s^2+6s+8}}{(s+4)(s-1)(s-2)} = \frac{s^2+6s+9}{(s+4)(s-1)(s-2)} = \frac{A}{s+4} + \frac{B}{s-1} + \frac{C}{s-2}$$

jiný způsob:

$$A = \frac{s^2+6s+9}{(s-1)(s-2)}\bigg|_{s=-4} = \frac{1}{(-5)(-6)} = \frac{1}{30}$$

$$B_{(s=1)} = \frac{s^2+6s+9}{(s+4)(s-2)}\bigg|_{s=1} = \frac{16}{5\cdot(-1)} = -\frac{16}{5}$$

$$C_{(s=2)} = \frac{s^2+6s+9}{(s-1)(s+4)}\bigg|_{s=2} = \frac{25}{6\cdot 1} = \frac{25}{6}$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \frac{1}{30}\mathcal{L}^{-1}\left\{\frac{1}{s+4}\right\} - \frac{16}{5}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + \frac{25}{6}\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} = \underline{\frac{1}{30}e^{-4t} - \frac{16}{5}e^{t} + \frac{25}{6}e^{2t}}$$

Laplaceova transformace derivací

① $\mathcal{L}\{f'(t)\} = \int_0^\infty e^{-pt} \cdot f'(t)\,dt = \left[e^{-pt} \cdot f(t)\right]_0^\infty + p\underbrace{\int_0^\infty e^{-pt} f(t)}_{\mathcal{L}\{f(t)\}} = 0 - f(0) + p \cdot \mathcal{L}\{f(t)\} \ominus$

$u = e^{-pt} \quad u' = -pe^{-pt}$

$v' = f'(t) \quad v = f(t)$

$\ominus -f(0) + p\,\underbrace{\mathcal{L}\{f(t)\}}_{=F(p)} = p \cdot F(p) - f(0)$

② $\mathcal{L}\{f''(t)\} = \int_0^\infty e^{-pt} \cdot f''(t)\,dt = \left[e^{-pt} \cdot f'(t)\right]_0^\infty + p\underbrace{\int_0^\infty e^{-pt} \cdot f'(t)\,dt}_{\mathcal{L}\{f'(t)\}} = 0 - f'(0) + p\,\mathcal{L}\{f'(t)\} \ominus$

$u = e^{-pt} \quad u' = -pe^{-pt}$

$v' = f''(t) \quad v = f'(t)$

$\ominus -f'(0) + p\left[p\,\mathcal{L}\{f(t)\} - f(0)\right] = -f'(0) - p\,f(0) + p^2\mathcal{L}\{f(t)\} =$

$= p^2 F(p) - p\,f(0) - f'(0)$

$\Rightarrow$ ③ $\mathcal{L}\{f'''(t)\} = p^3 F(p) - p^2 f(0) - s f'(0) - f''(0)$

$\rightarrow$ ⓝ $\mathcal{L}\{f^{(n)}(t)\} = p^n F(p) - s^{n-1} f(0) - s^{n-2} f'(0) - \ldots - f^{(n-1)}(0)$

Přímá Laplaceova transformace

Př:

a) $\mathcal{L}\{3t^2 - 5t + 1\} = 3\mathcal{L}\{t^2\} - 5\mathcal{L}\{t\} + \mathcal{L}\{1\} = 3\,\frac{2!}{p^3} - 5 \cdot \frac{1}{p^2} + \frac{1}{p} =$

$= \frac{6}{p^3} - \frac{5}{p^2} + \frac{1}{p}$

b) $\mathcal{L}\{2e^t - 3e^{-2t} + 5e^{-t}\} = 2\mathcal{L}\{e^t\} - 3\mathcal{L}\{e^{-2t}\} + 5\mathcal{L}\{e^{-t}\} = 2 \cdot \frac{1}{p-1} - 3\,\frac{1}{p+2} +$

$+ 5 \cdot \frac{1}{p+1} = \frac{2}{p-1} - \frac{3}{p+2} + \frac{5}{p+1}$

c) $\mathcal{L}\{te^{-2t} + t^2 e^{-3t}\} = \mathcal{L}\{te^{-2t}\} + \mathcal{L}\{t^2 e^{-3t}\} = \frac{1!}{(p+2)^2} + \frac{2!}{(p+3)^3} =$

$= \frac{1}{(p+2)^2} + \frac{2}{(p+3)^3}$

d) $\mathcal{L}\{3\sin 2t + 4\cos 2t\} = 3\mathcal{L}\{\sin 2t\} + 4\mathcal{L}\{\cos 2t\} = 3 \cdot \frac{2}{p^2+4} + 4 \cdot \frac{p}{p^2+4} = \frac{6}{p^2+4} + \frac{4p}{p^2+4} =$

$= \frac{6+4p}{p^2+4}$

DOPLNIT

$$\mathcal{L}\{t\cdot f(t)\} = -F'(p)$$

e) $$\mathcal{L}\{t\sin 2t + t\cos 5t\} = \mathcal{L}\{t\cdot\sin 2t\} + \mathcal{L}\{t\cos 5t\} = -\left(\frac{2}{p^2+4}\right)' - \left(\frac{p}{p^2+25}\right)' =$$

$$= -\frac{-2\cdot 2p}{(p^2+4)^2} - \frac{1(p^2+25) - p\cdot 2p}{(p^2+25)^2} = \frac{4p}{(p^2+4)^2} + \frac{-p^2-25+2p^2}{(p^2+25)^2} =$$

$$= \frac{4p}{(p^2+4)^2} + \frac{p^2-25}{(p^2+25)^2}$$

f) $$\mathcal{L}\{e^{-3t}\sin 2t + e^{-2t}\cos 2t\} = \mathcal{L}\{e^{-3t}\sin 2t\} + \mathcal{L}\{e^{-2t}\cos 2t\} =$$

$$= \frac{2}{(p+3)^2+4} + \frac{p+2}{(p+2)^2+4} = \frac{2}{p^2+6p+13} + \frac{p+2}{p^2+4p+8}$$

cvičení

① $\mathcal{L}\{2+3te^{-2t}-4t^2e^{-3t}\} = \mathcal{L}\{2\}+3\mathcal{L}\{te^{-2t}\}-4\mathcal{L}\{t^2e^{-3t}\} =$

$= \frac{2}{p} + 3\cdot\frac{1}{(p+2)^2} - 4\cdot\frac{2}{(p+3)^2}$

② $\mathcal{L}\{3\sin 2t - 5\cos 2t\} = 3\mathcal{L}\{\sin 2t\} - 5\mathcal{L}\{\cos 2t\} = 3\cdot\frac{2}{p^2+4} - 5\cdot\frac{p}{p^2+4} = \frac{6-5p}{p^2+4}$

③ $\mathcal{L}\{3t - \sin 2t\} = \frac{3}{p^2} - \frac{2}{p^2+4}$

④ $\mathcal{L}\{(2t+5)e^{-2t} + 3\cos t - 2\sin 3t\} = 2\mathcal{L}\{te^{-2t}\} + 5\mathcal{L}\{e^{-2t}\} + 3\mathcal{L}\{\cos t\} - 2\mathcal{L}\{\sin 3t\} \ominus$

doplnit: $\mathcal{L}\{e^{-at}f(t)\} = F(p+a)$

$\ominus\ 2\cdot\frac{1}{(p+2)^2} + \frac{5}{p+2} + 3\frac{p}{p^2+1} - 2\frac{3}{p^2+9} = \frac{2}{(p+2)^2} + \frac{5}{p+2} + \frac{3p}{p^2+1} - \frac{6}{p^2+9}$

⑤ $\mathcal{L}\{(t+2)\cos 3t\} = \mathcal{L}\{t\cos 3t\} + 2\mathcal{L}\{\cos 3t\} = -\left(\frac{p}{p^2+9}\right)' + \frac{2p}{p^2+9} =$

$= -\frac{p^2+9-p(2p)}{(p^2+9)^2} + \frac{2p}{p^2+9} = \frac{p^2-9}{(p^2+9)^2} + \frac{2p}{p^2+9} = \frac{p^2-9+2p(p^2+9)}{(p^2+9)^2} = \frac{2p^3+p^2+18p-9}{(p^2+9)^2}$

⑥ $\mathcal{L}\{te^{-3t} - 2e^{-2t}\sin 3t + 4\} = \mathcal{L}\{te^{-3t}\} - 2\mathcal{L}\{e^{-2t}\sin 3t\} + \mathcal{L}\{4\} =$

$= \frac{1}{(p+3)^2} - 2\frac{3}{(p+2)^2+9} + \frac{4}{p} = \frac{1}{(p+3)^2} - \frac{6}{(p+2)^2+9} + \frac{4}{p}$

cvičení

Př. Pomocí tabulky nalezněte vzory daných obrazů $F(p)$

a) $F(p) = \dfrac{3}{p-8}$

$$\mathcal{L}^{-1}\{F(p)\} = \mathcal{L}^{-1}\left\{\frac{3}{p-8}\right\} = 3\,\mathcal{L}^{-1}\left\{\frac{1}{p-8}\right\} = \underline{\underline{3e^{8t}}}$$

b) $F(p) = \dfrac{p+1}{p^2+9}$

$$\mathcal{L}^{-1}\left\{\frac{p+1}{p^2+9}\right\} = \mathcal{L}^{-1}\left\{\frac{p}{p^2+9} + \frac{1}{p^2+9}\right\} = \mathcal{L}^{-1}\left\{\frac{p}{p^2+9}\right\} + \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{3}{p^2+9}\right\} =$$

$$= \underline{\underline{\cos 3t + \frac{1}{3}\sin 3t}}$$

c) $F(p) = \dfrac{p}{p^2+2p+2}$

$$\mathcal{L}^{-1}\left\{\frac{p+1-1}{(p+1)^2+1}\right\} = \mathcal{L}^{-1}\left\{\frac{p+1}{(p+1)^2+1}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{(p+1)^2+1}\right\} = \underline{\underline{e^{-t}\cos t - e^{-t}\sin t}}$$

d) $F(p) = \dfrac{2p}{p^2+4p+7}$

$$\mathcal{L}^{-1}\left\{\frac{2p}{p^2+4p+7}\right\} = 2\,\mathcal{L}^{-1}\left\{\frac{p-2+2}{(p+2)^2+3}\right\} = 2\,\mathcal{L}^{-1}\left\{\frac{p+2}{(p+2)^2+3}\right\} - 4\,\mathcal{L}^{-1}\left\{\frac{1}{(p+2)^2+3}\right\} =$$

$$= 2\,\mathcal{L}^{-1}\left\{\frac{p+2}{(p+2)^2+3}\right\} - \frac{4}{\sqrt{3}}\mathcal{L}^{-1}\left\{\frac{\sqrt{3}}{(p+2)^2+3}\right\} = \underline{\underline{2\cdot e^{-2t}\cos\sqrt{3}t - \frac{4}{\sqrt{3}}e^{-2t}\sin\sqrt{3}t}}$$

e) $F(p) = \left\{\dfrac{p+1}{p(p+2)}\right\}$

$$\mathcal{L}^{-1}\left\{\frac{p+1}{p(p+2)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{2p} + \frac{1}{2(p+2)}\right\} = \underline{\underline{\frac{1}{2}\cdot 1 + \frac{1}{2}e^{-2t}}}$$

$$\frac{A}{p} + \frac{B}{p+2}$$

$$Ap + 2A + Bp = p+1$$

$p:\ 1 = A + B \qquad B = \frac{1}{2}$

$p^0:\ 1 = 2A \qquad A = \frac{1}{2}$

Př: Pomocí vět o rozkladu najděte vzory daných obrazů $F(p)$ ②

a) $F(p) = \dfrac{p+2}{p^2-4p+3}$ $\qquad p^2-4p+3=0$

$(p-1)(p-3)=0 \quad p_1 = 1 \quad$ 1.ř

$p_2 = 3$

$$\mathcal{L}^{-1}\left\{\frac{p+2}{p^2-4p+3}\right\} = \operatorname*{res}_{p=1} F(p)e^{pt} + \operatorname*{res}_{p=3} F(p)e^{pt} = \lim_{p\to1} \cancel{(p-1)}\frac{p+2}{\cancel{(p-1)}(p-3)}e^{pt} +$$

$$+\lim_{p\to3} \cancel{(p-3)}\frac{p+2}{(p-1)\cancel{(p-3)}}e^{pt} = -\frac{3}{2}e^{t} + \frac{5}{2}e^{3t}$$

b) $F(p) = \dfrac{p^2+1}{(p+1)^2(p-1)}$ $\qquad (p+1)^2(p-1)=0$

$p_{12} = -1 \quad p_3 = 1$

2.ř $\quad$ 1.ř

$$\mathcal{L}^{-1}\left\{\frac{p^2+1}{(p+1)^2(p-1)}\right\} = \operatorname*{res}_{p=-1} F(p)e^{pt} + \operatorname*{res}_{p=1} F(p)e^{pt} = \lim_{p\to-1}\left[\cancel{(p+1)^2}\frac{p^2+1}{\cancel{(p+1)^2}(p-1)}e^{pt}\right]' +$$

$$+\lim_{p\to1}\left[\cancel{(p-1)}\frac{p^2+1}{(p+1)^2\cancel{(p-1)}}e^{pt}\right] = \lim_{p\to1}\frac{(p^2+1)e^{pt}}{(p+1)^2} + \lim_{p\to-1}\frac{[2p(e^{pt})+(p^2+1)te^{pt}](p-1)-(p^2+1)e^{pt}}{(p-1)^2}$$

$$= \frac{1}{2}e^{t} - \frac{4te^{-t}}{4} + \frac{4e^{-t}}{4} - \frac{2e^{-t}}{4} = \frac{1}{2}e^{t} - te^{-t} + e^{-t} - \frac{1}{2}e^{-t} = \frac{1}{2}e^{t} - te^{-t} + \frac{1}{2}e^{-t}$$

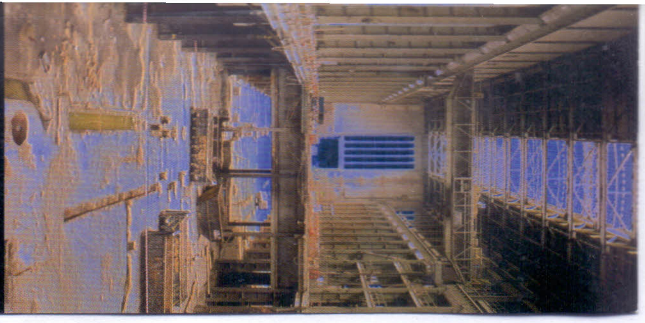
c) $F(p) = \dfrac{p^2+p-1}{(p-2)(p^2-p-20)} = \dfrac{p^2+p-1}{(p-2)(p-5)(p+4)}$ $\qquad$ póly: $p_1=2,\ p_2=5,\ p_3=-4$ 1.ř

$$\mathcal{L}^{-1}\left\{\frac{p^2+p-1}{(p-2)(p^2-p+20)}\right\} = \operatorname*{res}_{p=2} F(p)e^{pt} + \operatorname*{res}_{p=5} F(p)e^{pt} + \operatorname*{res}_{p=-4} F(p)e^{pt} =$$

$$= \lim_{p\to2}\left[\cancel{(p-2)}\frac{p^2+p-1}{\cancel{(p-2)}(p-5)(p+4)}e^{pt}\right] + \lim_{p\to5}\left[\cancel{(p-5)}\frac{p^2+p-1}{(p-2)\cancel{(p-5)}(p+4)}e^{pt}\right] +$$

$$+\lim_{p\to-4}\left[\cancel{(p+4)}\frac{p^2+p-1}{(p-2)(p-5)\cancel{(p+4)}}e^{pt}\right] = \frac{5}{(-3)6}e^{2t} + \frac{29}{3\cdot9}e^{5t} + \frac{11}{(-6)(-9)}e^{-4t} =$$

$$= -\frac{5}{18}e^{2t} + \frac{29}{27}e^{5t} + \frac{11}{54}e^{-4t}$$



Př: Naleznite vzory daných obrazů $F(p)$

a) $F(p) = \dfrac{p^2+4}{p^3+p^2-2p}$ póly: $p(p^2+p-2)$

$p(p+2)(p-1)=0$

$p_1=0 \quad \underbrace{p_2=-2 \quad p_3=1}_{1.ř.}$

$$\mathcal{L}^{-1}\left\{\frac{p^2+4}{p^3+p^2-2p}\right\} = \operatorname*{res}_{p=0} F(p)e^{pt} + \operatorname*{res}_{p=-2} F(p)e^{pt} + \operatorname*{res}_{p=1} F(p)e^{pt} =$$

$$= \lim_{p\to 0}\left[\cancel{(p-0)}\frac{p^2+4}{\cancel{p}(p+2)(p-1)}\cdot e^{pt}\right] + \lim_{p\to -2}\left[\cancel{(p+2)}\frac{p^2+4}{p\cancel{(p+2)}(p-1)}e^{pt}\right] + \lim_{p\to 1}\left[\cancel{(p-1)}\frac{(p^2+4)e^{pt}}{p(p+2)\cancel{(p-1)}}\right] =$$

$$= \frac{4}{-2}e^0 + \frac{8}{(-2)(-3)}e^{-2t} + \frac{5}{3}e^t = \underline{\underline{-2 + \frac{4}{3}e^{-2t} + \frac{5}{3}e^t}}$$

b) $F(p) = \dfrac{4}{(p+1)^4} - \dfrac{p}{p^2+2}$

$$\mathcal{L}^{-1}\left\{\frac{4}{(p+1)^4} - \frac{p}{p^2+2}\right\} = \frac{4}{6}\mathcal{L}^{-1}\left\{\frac{6}{(p+1)^4}\right\} - \mathcal{L}^{-1}\left\{\frac{p}{p^2+2}\right\} =$$

$$= \underline{\underline{\frac{2}{3}t^3e^{-t} - \cos\sqrt{2}t}}$$

c) $F(p) = \dfrac{2p}{(p^2+1)(p^2+4)}$ $\dfrac{Ap+B}{p^2+1} + \dfrac{Cp+D}{p^2+4} = \dfrac{(Ap+B)(p^2+4)+(Cp+D)(p^2+1)}{(p^2+1)(p^2+4)} =$

$$= \frac{Ap^3+4Ap+Bp^2+4B+Cp^3+Cp+Dp^2+D}{(p^2+1)(p^2+4)}$$

$p^3: 0 = A + C$
$p^2: 0 = B + D$
$p: 2 = 4A + C$
$p^0: 0 = 4B + D$

$0 = A + C$
$2 = 4A + C$
$-2 = -3A \quad A = \frac{2}{3}$
$C = -\frac{2}{3}$

$0 = B + D$
$0 = 4B + D$
$0 = -3B \quad B = 0 \quad D = 0$

$$\frac{2p}{(p^2+1)(p^2+4)} = \frac{2p}{3(p^2+1)} - \frac{2p}{3(p^2+4)}$$

$$\mathcal{L}^{-1}\left\{\frac{2p}{(p^2+1)(p^2+4)}\right\} = \frac{2}{3}\mathcal{L}^{-1}\left\{\frac{p}{p^2+1}\right\} - \frac{2}{3}\mathcal{L}^{-1}\left\{\frac{p}{p^2+4}\right\} = \underline{\underline{\frac{2}{3}\cos t - \frac{2}{3}\cos 2t}}$$

d) $F(p) = \frac{1}{(p+3)(p-2)^2}$ póly $\underset{1.r}{p_1 = -3}$ $\underset{2.r}{p_{23} = 2}$

$$\mathcal{L}^{-1}\left\{\frac{1}{(p+3)(p-2)^2}\right\} = \operatorname*{res}_{p=-3} F(p)e^{pt} + \operatorname*{res}_{p=2} F(p)e^{pt} =$$

$$= \lim_{p\to-3}\left[\cancel{(p+3)}\frac{1}{\cancel{(p+3)}(p-2)^2}e^{pt}\right] + \lim_{p\to 2}\left[(p-2)^2\frac{1}{(p+3)(p-2)^2}e^{pt}\right]' =$$

$$= \frac{1}{25}e^{-3t} + \lim_{p\to2}\frac{te^{pt}(p+3)-e^{pt}}{(p+3)^2} = \frac{1}{25}e^{-3t}+\frac{5te^{2t}-e^{2t}}{25} = \frac{1}{25}\left[e^{-3t}+e^{2t}(5t-1)\right]$$

e) $F(p)=\frac{1}{(p-1)^2(p-2)^3}$ póly $\underset{2.r}{p_{12}=1}$ $\underset{3.r}{p_{345}=2}$

$$\mathcal{L}^{-1}\left\{\frac{1}{(p-1)^2(p-2)^3}\right\} = \operatorname*{res}_{p=1} F(p)e^{pt} + \operatorname*{res}_{p=2}F(p)e^{pt} =$$

$$\operatorname*{res}_{p=1} = \lim_{p\to1}\left[\cancel{(p-1)^2}\frac{1}{\cancel{(p-1)^2}(p-2)^3}e^{pt}\right]' = \lim_{p\to1}\left[(p-2)^{-3}e^{pt}\right]' = \lim_{p\to1}\left[(-3)(p-2)^{-4}e^{pt}+t(p-2)^{-3}e^{pt}\right]=$$

$$=\lim_{p\to1}\left[\frac{-3}{(p-2)^4}e^{pt}+\frac{t}{(p-2)^3}e^{pt}\right] = -3e^t - te^t$$

$$\operatorname*{res}_{p=2} = \frac12\lim_{p\to2}\left[\cancel{(p-2)^3}\frac{1}{(p-1)^2\cancel{(p-2)^3}}e^{pt}\right]'' = \frac12\lim_{p\to2}\left[(p-1)^{-2}e^{pt}\right]''=$$

$$=\frac12\lim_{p\to2}\left[(-2)(p-1)^{-3}e^{pt}+(p-1)^{-2}te^{pt}\right]' = \frac12\lim_{p\to2}\left\{(-2)\left[(-3)(p-1)^{-4}e^{pt}+(p-1)^{-3}te^{pt}\right]+\right.$$

$$\left.+t\left[(-2)(p-1)^{-3}e^{pt}+(p-1)^{-2}t\cdot e^{pt}\right]\right\} = \frac12\lim_{p\to2}\left\{\frac{6}{(p-1)^4}e^{pt}-\frac{2t}{(p-1)^3}e^{pt}-\frac{2t}{(p-1)^3}e^{pt}+\frac{t^2}{(p-1)^2}e^{pt}\right\}=$$

$$=\frac12\left(6e^{2t}-4te^{2t}+t^2e^{2t}\right) = 3e^{2t}-2te^{2t}+\frac12t^2e^{2t}$$

$$= -3e^t - te^t+3e^{2t}-2te^{2t}+\frac12t^2e^{2t}$$

f) $F(p)=\frac{p^2+1}{p(p+2)(p+1)}$ póly $\underset{1.r}{p_1=0}$ $\underset{1.r}{p_2=-2}$ $\underset{1.r}{p_3=-1}$

$$\mathcal{L}^{-1}\left\{\frac{p^2+1}{p(p+2)(p+1)}\right\} = \operatorname*{res}_{p=0} F(p)e^{pt} + \operatorname*{res}_{p=-2} F(p)e^{pt} + \operatorname*{res}_{p=-1} F(p)e^{pt} = \lim_{p\to0}\left[\cancel{p}\frac{p^2+1}{\cancel{p}(p+2)(p+1)}e^{pt}\right]+$$

$$+\lim_{p\to-2}\left[\cancel{(p+2)}\frac{p^2+1}{p\cancel{(p+2)}(p+1)}e^{pt}\right]+\lim_{p\to-1}\left[\cancel{(p+1)}\frac{p^2+1}{p(p+2)\cancel{(p+1)}}e^{pt}\right]=$$

$$=\frac12e^0+\frac52e^{-2t}-2e^{-t}$$