1

Fourierovy Řady

- Joseph Fourier (1768-1830) řešil problematiku vedení tepla a potřeboval vyjádřit funkci $f(x)$ jako nekonečnou řadu sinů a kosinů

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) =$$

$$= a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + a_3 \cos 3x + b_3 \sin 3nx + \cdots \qquad (1)$$

- Dříve už Daniel Bernoulli a Leonard Euler používali tuto řadu při zkoumání problému kmitání struny a v astronomii
- Řada v rovnici (1) se nazývá Trigonometrická Řada (TŘ) nebo Fourierova řady
- Vyjádření funkce ve tvaru TŘ je v některých aplikacích vhodnější než Taylorova mocninná řada (hlavně v případech periodických jevů – kmitání apod.)
- Začínáme předpokladem, že TŘ konverguje a spojitá funkce $f(x)$ je jejím součtem na intervalu $\langle -\pi; \pi\rangle$ tak, že

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx), \quad -\pi \le x \le \pi. \qquad (2)$$

- Náš úkol je nalézt vztahy pro koeficienty a_0, a_n a b_n v závislosti na $f(x)$.

PŘ1 PŘ2 PŘ3

- V příkladech 1, 2 a 3 jsme odvodili vztahy:
- $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx$, $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx$, $n = 1, 2, 3, \cdots$, a
- $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx$, $n = 1, 2, 3, \cdots$.
- Předpokládali jsme, že $f(x)$ je spojitá.

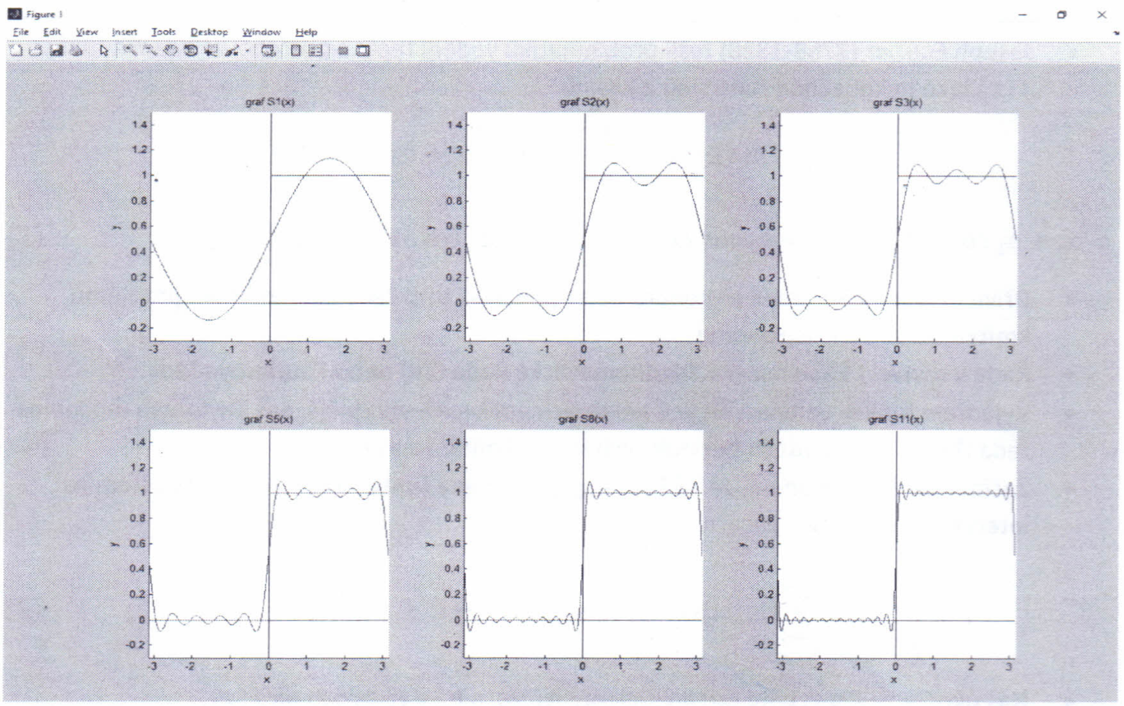
Nyní definici rozšíříme a budeme požadovat, jen aby funkce $f(x)$ byla alespoň po částech spojitá na intervalu $\langle -\pi; \pi\rangle$, kromě konečného počtu bodů, kde spojitá není.

Definice:
Nechť funkce $f(x)$ je po částech spojitá na intervalu $\langle -\pi; \pi\rangle$. Potom *Fourierova řada funkce* $f(x)$ je řada $a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$, kde koeficienty a_0, a_n a b_n jsou definovány:

$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx$, $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx$ a $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx$ a jsou nazýváni Fourierovými koeficienty funkce $f(x)$.

PŘ4

Obrázek k příkladu 5:



- Na obrázku vidíme, že se zvětšujícím se n se $S_n(x)$ stává stále lepší aproximací grafu $f(x)$. Vypadá to, že graf $S_n(x)$ se blíží grafu $f(x)$, kromě bodu 0 nebo $\pm\pi$. Jinými slovy to vypadá, jako že funkce $f(x)$ se rovná součtu její Fourierovy řady kromě bodů, kde není funkce $f(x)$ spojitá.
- Toto je typické pro po částech spojitou funkci, která má konečný počet bodů nespojitosti
- V bodě a, kde není funkce $f(x)$ spojitá existují jednostranné limity: $\lim\limits_{x\to a^+} f(x) = f(x+)$, resp. $\lim\limits_{x\to a^-} f(x) = f(x-)$.

Fourierův konvergenční teorém (FKT):

Je-li $f(x)$ periodická funkce s periodou 2π a $f(x)$ a $f'(x)$ jsou po částech spojité na intervalu $\langle -\pi; \pi\rangle$, potom je Fourierova řada konvergentní. Součet Fourierovy řady je roven funkci $f(x)$ ve všech bodech, kde je funkce $f(x)$spojitá. Tam, kde funkce $f(x)$ není spojitá, je součet Fourierovy řady roven aritmetickému průměru limity zleva a zprava $\frac{1}{2}[f(x+) + f(x-)]$

- Pokud aplikujeme FKT na funkci z příkladu 5 (čtvercová vlna) dostaneme:
- $f(x+) = 1, f(x-) = 0$.
- Průměrná hodnota je tedy $\frac{1}{2}$. Stejné je to i pro body nespojitosti $n\pi$.

$$\forall n \in N;\ f(x) = \begin{cases} \dfrac{1}{2} + \displaystyle\sum_{k=1}^{\infty} \frac{2}{(2k-1)\pi} \sin(2k-1)x, pro\ x \neq n\pi \\ \dfrac{1}{2}, pro\ x = n\pi \end{cases}$$

PKS

Funkce s periodou $2L$

- Pokud je funkce $f(x)$ periodická, ale perioda není 2π, můžeme nalézt Fourierovu řadu pomocí substituce
- Předpokládejme, že $f(x)$ má periodu $2L$, tak, že $f(x+2L)=f(x)\ \forall x$
- Zavedeme substituci $t=\frac{\pi x}{L}$ a dostaneme $g(t)=f(x)=f(\frac{Lt}{\pi})$
- Potom funkce $g(t)$ má periodu 2π a $x=\pm L$ odpovídá $t=\pm\pi$.

Fourierova řada funkce $g(t)$ je: $g(t)=a_0+\sum_{n=1}^{\infty}(a_n\cos nt+b_n\sin nt)$ a $a_0=\frac{1}{2\pi}\int_{-\pi}^{\pi}g(t)dt$, $a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}g(t)\cos nt\,dt$, a $b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}g(t)\sin nt\,dt$.

- Pokud nyní použijeme substituci $x=\frac{Lt}{\pi}$, potom $t=\frac{\pi x}{L}$, $dt=\frac{\pi}{L}dx$ dostaneme:

Jestliže $f(x)$ je po částech spojitá funkce na intervalu $\langle -L;L\rangle$, pak je její Fourierova řada

$$f(x)=a_0+\sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\pi x}{L}\right)+b_n\sin\left(\frac{n\pi x}{L}\right)\right],$$

kde

$a_0=\frac{1}{2L}\int_{-L}^{L}f(t)dx,$

a pro $n\geq 1$,

$$a_n=\frac{1}{L}\int_{-L}^{L}f(x)\cos\left(\frac{n\pi x}{L}\right)dx,\qquad b_n=\frac{1}{L}\int_{-L}^{L}f(x)\sin\left(\frac{n\pi x}{L}\right)dx.$$

- FKT platí i pro periodické funkce s periodou $2L$.

PŘ 6

Fourierovy Řady (FŘ) - shrnutí

- Reálnou funkci na konečném intervalu vyjádříme ve tvaru nekonečné řady, která se skládá ze sinů a kosinů (tvar FŘ)

Definice a vlastnosti FŘ

Nekonečná řada tvaru $\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos n\omega t + b_n \sin n\omega t)$, kde $\omega = \frac{2\pi}{T}$, se nazývá *Trigonometrickou řadou* s periodou T.

Nechť $T > 0$ a $f: \langle a; a+T\rangle \to R$

- je reálná funkce reálné proměnné
- je na tomto intervalu po částech spojitá
- má po částech spojitou derivaci (=po částech hladká),

potom můžeme k funkci f sestrojit trigonometrickou řadu s periodou T, ve které pro čísla a_n a b_n platí: $a_n = \frac{2}{T}\int_a^{a+T} f(t)\cos n\omega t\, dt$; $b_n = \frac{2}{T}\int_a^{a+T} f(t)\sin n\omega t\, dt$.

Tuto řadu nazýváme *Fourierovou řadou* funkce f a píšeme:

$$f(t) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos n\omega t + b_n \sin n\omega t)\,, \omega = \frac{2\pi}{T}$$

čísla a_n a b_n se nazývají *Fourierovy koeficienty*

- Součet FŘ je roven zadané funkci tam, kde je funkce spojitá:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos n\omega t + b_n \sin n\omega t)\,, t \in (a; a+T)$$

- V bodech nespojitosti se součet rovná aritmetickému průměru limity zprava a limity zleva:

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos n\omega t + b_n \sin n\omega t) = \frac{1}{2}[f(t)^+ + f(t)^-].$$

Nechť funkce $f(t)$ je po částech hladká na symetrickém intervalu $\langle -\frac{T}{2}; \frac{T}{2}\rangle$. Potom:

- Je-li SUDÁ, pak $b_n = 0, a_n = \frac{4}{T}\int_0^{\frac{T}{2}} f(t)\cos n\omega t\, dt;\ \omega = \frac{2\pi}{T}$
  - Fourierův rozvoj funkce se zredukuje na kosinovou řadu
- Je-li LICHÁ, pak $a_n = 0, b_n = \frac{4}{T}\int_0^{\frac{T}{2}} f(t)\sin n\omega t\, dt;\ \omega = \frac{2\pi}{T}$
  - Fourierův rozvoj funkce se zredukuje na sinovou řadu.

Př 7 Př 8

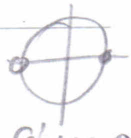
$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

PR1 hledáme koef. a_0 — integrujeme (2) přes interval $\langle -\pi, \pi \rangle$

$$\int_{-\pi}^{\pi} f(x)\,dx = \int_{-\pi}^{\pi} a_0\,dx + \int_{-\pi}^{\pi}\sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)\,dx = a_0[x]_{-\pi}^{\pi} +$$

$$+ \sum_{n=1}^{\infty} a_n \underbrace{\int_{-\pi}^{\pi} \cos nx\,dx}_{0} + \sum_{n=1}^{\infty} b_n \underbrace{\int_{-\pi}^{\pi} \sin nx\,dx}_{0} = a_0[\pi-(-\pi)] + 0 = a_0 2\pi$$

$$\int_{-\pi}^{\pi} \cos nx\,dx = \frac{1}{n}[\sin nx]_{-\pi}^{\pi} = \frac{1}{n}[\sin n\pi - \underbrace{\sin(-n\pi)}_{+\sin(n\pi)}] = \frac{2}{n}(\underbrace{\sin n\pi}_{0}) = 0$$



$\sin n\pi = 0$

$$\int_{-\pi}^{\pi} \sin nx\,dx = -\frac{1}{n}[\cos nx]_{-\pi}^{\pi} = -\frac{1}{n}[\cos n\pi - \underbrace{\cos(-n\pi)}_{-\cos n\pi}] = -\frac{1}{n}\cdot 0 = 0$$

tedy $\int_{-\pi}^{\pi} f(x)\,dx = a_0 \cdot 2\pi \Rightarrow a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$

PR2 hledáme koef. a_n $n \geq 1$ — násobíme obě strany rce (2) $\cos mx$ $(m \in N, m \geq 1)$

$$f(x)\cdot \cos mx\,dx = \left[a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)\right]\cdot \cos mx$$

dále opět integrujeme přes interval $\langle -\pi, \pi \rangle$

$$\int_{-\pi}^{\pi} f(x)\cos mx\,dx = a_0 \underbrace{\int_{-\pi}^{\pi} \cos mx\,dx}_{\substack{0 \\ \text{viz PR1}}} + \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} \cos nx \cdot \cos mx\,dx + \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi} \sin nx \cdot \cos mx\,dx$$

$\boxed{\text{rozdělíme a) } m \neq n}$

$$\int_{-\pi}^{\pi} \cos nx \cos mx\,dx = \frac{m^2}{m^2-n^2}\left[\frac{1}{m}\cos nx \sin mx - \frac{n}{m^2}\sin nx \cos mx\right]_{-\pi}^{\pi} \ominus$$

$$\int \cos nx \cos mx\,dx = \frac{1}{m}\cos nx \cdot \sin mx + \frac{n}{m}\int \sin nx \sin mx\,dx = \frac{1}{m}\cos nx \sin mx \oplus$$

$u = \cos nx \quad u' = -n\sin nx$

$v' = \cos mx \quad v = \frac{1}{m}\sin mx$

$u = \sin nx \quad u' = n\cos nx$

$v' = \sin mx \quad v = -\frac{1}{m}\cos mx$

$$\oplus\ \frac{n}{m}\left(-\frac{1}{m}\sin nx \cos mx + \frac{n}{m}\int \cos nx \cos mx\,dx\right)$$

$$\Rightarrow \int \cos nx \cos mx\,dx = \frac{1}{m}\cos nx \sin mx - \frac{n}{m^2}\sin nx \cos mx + \frac{n^2}{m^2}\int \cos nx \cos mx$$

$$\left(1-\frac{n^2}{m^2}\right)\int \cos nx \cos mx\,dx = \frac{1}{m}\cos nx \sin mx - \frac{n}{m^2}\sin nx \cos mx$$

$$\int \cos nx \cos mx\,dx = \frac{m^2}{m^2-n^2}\left(\frac{1}{m}\cos nx \sin mx - \frac{n}{m^2}\sin nx \cos mx\right)$$

$$= \frac{m^2}{m^2-n^2}\left[\left(\frac{1}{m}\cos\pi n \sin m\pi - \frac{n}{m^2}\sin n\pi \cos m\pi\right) - \left(\frac{1}{m}\cos(-n\pi)\sin(-m\pi) - \frac{n}{m^2}\sin(-n\pi)\cdot\cos(-m\pi)\right)\right] =$$

$$= \frac{m^2}{m^2-n^2}[0] = 0$$

$$\int_{-\pi}^{\pi}\sin nx \cos mx\,dx = \frac{m^2}{m^2-n^2}\left[\frac{1}{m}\sin nx \sin mx + \frac{n^2}{m^2}\cos nx \cos mx\right]_{-\pi}^{\pi} =$$

$$\int \sin nx \cos mx\,dx = \frac{1}{m}\sin nx \sin mx - \frac{n}{m}\int \cos nx \sin mx\,dx =$$

$u = \sin nx \quad u' = n\cos nx$ $\qquad$ $u = \cos nx \quad u' = -n\sin nx$

$v' = \cos mx \quad v = \frac{1}{m}\sin mx$ $\qquad$ $v' = \sin mx \quad v = -\frac{1}{m}\cos mx$

$$= \frac{1}{m}\sin nx \sin mx - \frac{n}{m}\left(-\frac{1}{m}\cos nx \cos mx - \frac{n}{m}\int \sin nx \cos mx\,dx\right) =$$

$$= \frac{1}{m}\sin nx \sin mx + \frac{n}{m^2}\cos nx \cos mx + \frac{n^2}{m^2}\int \sin nx \cos mx\,dx)$$

$$\Rightarrow \left(1-\frac{n^2}{m^2}\right)\int \sin nx \cos mx\,dx = \frac{1}{m}\sin nx \sin mx + \frac{n^2}{m^2}\cos nx \cos mx$$

$$\int \sin nx \cos mx\,dx = \frac{m^2}{m^2-n^2}\left(\frac{1}{m}\sin nx \sin mx + \frac{n^2}{m^2}\cos nx \cos mx\right)$$

$$= \frac{m^2}{m^2-n^2}\left[\left(\frac{1}{m}\sin n\pi \sin m\pi + \frac{n^2}{m^2}\cos n\pi \cos m\pi\right) - \left(\frac{1}{m}\sin(-n\pi)\sin(-m\pi) + \frac{n^2}{m^2}\cos(-n\pi)\cdot\cos(-m\pi)\right)\right] =$$

$$= \frac{m^2}{m^2-n^2}\left(\frac{n^2}{m^2}(-1)^n\cdot(-1)^m - \frac{n^2}{m^2}(-1)^n\cdot(-1)^m\right) = \frac{m^2}{m^2-n^2}\cdot\frac{n^2}{m^2}\left(\underbrace{(-1)^{n+m} - (-1)^{n+m}}_{0}\right) = 0$$

b) $m = n$

$$\int_{-\pi}^{\pi}\cos nx \cos nx = \int_{-\pi}^{\pi}\cos^2 nx\,dx = \int_{-\pi}^{\pi}\frac{1+\cos 2nx}{2}\,dx = \int_{-\pi}^{\pi}\left(\frac{1}{2}+\frac{1}{2}\cos 2nx\right)dx =$$

$$= \left[\frac{1}{2}x + \frac{1}{2}\cdot\frac{1}{2n}\sin 2nx\right]_{-\pi}^{\pi} = \left[\left(\frac{1}{2}\pi + \underbrace{\frac{1}{4n}\sin 2n\pi}_{0}\right) - \left(-\frac{1}{2}\pi + \underbrace{\frac{1}{4n}\sin(-2n\pi)}_{0}\right)\right] = \frac{\pi}{2}+\frac{\pi}{2} = \pi$$

$$\int_{-\pi}^{\pi}\sin nx \cos nx\,dx = \frac{1}{2}\int_{-\pi}^{\pi}2\sin nx \cos nx\,dx = \frac{1}{2}\int_{-\pi}^{\pi}\sin 2nx\,dx = -\frac{1}{2}\cdot\frac{1}{2n}[\cos 2nx]_{-\pi}^{\pi} =$$

$$= -\frac{1}{4n}\left[\underbrace{\cos 2n\pi - \underbrace{\cos(2n\pi)}_{-\cos 2n\pi}}_{0}\right] = 0$$

Dosadíme do Φ

$$\int_{-\pi}^{\pi} f(x)\cos mx\,dx = a_0\underbrace{\int_{-\pi}^{\pi}\cos mx\,dx}_{0} + \sum_{n=1}^{\infty} a_n \underbrace{\int_{-\pi}^{\pi}\cos nx\cos mx\,dx}_{\pi} + \sum_{n=1}^{\infty} b_n \underbrace{\int_{-\pi}^{\pi}\sin nx\cos mx\,dx}_{0}$$

pro $m \neq n$ vyšly 0

pro $n=m$ jen $\sum_{n=1}^{\infty} a_n$ (pro $m=n$) je tam jen jeden $= a_n$

$$\int_{-\pi}^{\pi} f(x)\cos mx\,dx = a_n \cdot \pi \Rightarrow a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cdot\cos mx\,dx$$

Př 3 Hledáme koef b_n $n \geq 1$ násobíme obě strany rce (2) $\sin mx$ ($m \in N$; $m \geq 1$)

$$f(x)\sin mx\,dx = \left[a_0 + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)\right]\cdot\sin mx$$

dále opět integrujeme přes interval $\langle -\pi, \pi\rangle$

$$\int_{-\pi}^{\pi} f(x)\sin nx\,dx = a_0\underbrace{\int_{-\pi}^{\pi}\sin mx\,dx}_{0} + \sum_{n=1}^{\infty} a_n \underbrace{\int_{-\pi}^{\pi}\cos nx\sin mx\,dx}_{0} + \sum_{n=1}^{\infty} b_n \underbrace{\int_{-\pi}^{\pi}\sin nx\cdot\sin mx\,dx}_{\pi}$$

podobně pro $m \neq n$ jsou $\int = 0$

pro $m=n$ $\quad 0 \quad \pi$

jen jeden b_n ze $\sum_{n=1}^{\infty} b_n$ $n=m$!

$$\int_{-\pi}^{\pi} f(x)\sin nx\,dx = b_n\cdot\pi \Rightarrow b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

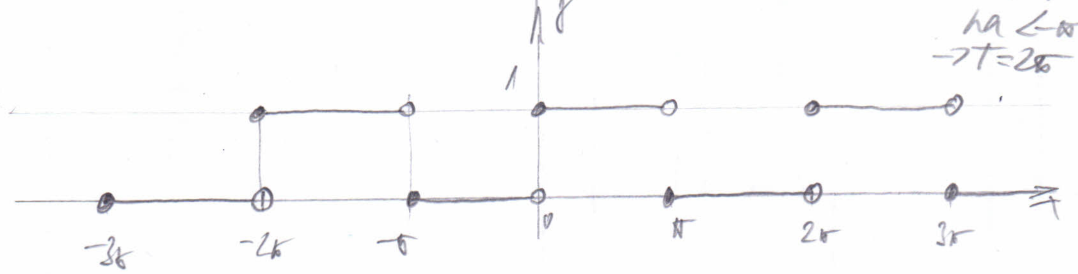
Př 4 Nalezněte Fourierovy koeficienty a Fourierovu řadu pro funkci obdélníkové vlny definované

$$f(x) = \begin{cases} 0 & \text{když } -\pi \leq x < 0 \\ 1 & \text{když } 0 \leq x < \pi \end{cases} \quad \text{a } f(x+2\pi) = f(x)$$

→ fu je po částech spojitá na $\langle -\pi, \pi\rangle$

→ $T = 2\pi$ $\omega = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1$

graf:



$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\int_{-\pi}^{0} 0\,dx + \frac{1}{2\pi}\int_{0}^{\pi} 1\,dx = 0 + \frac{1}{2\pi}[\pi - 0] = \frac{\pi}{2\pi} = \frac{1}{2}$$

pro $n \geq 1$ $$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{0} 0\,dx + \frac{1}{\pi}\int_{0}^{\pi} \cos nx\,dx = 0 + \frac{1}{\pi}\cdot\frac{1}{n}\left[\sin nx\right]_0^{\pi} =$$

$$= 0 + \frac{1}{\pi n}\left[\underbrace{\sin n\pi}_{0} - \underbrace{\sin 0}_{0}\right] = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{0} 0\,dx + \frac{1}{\pi}\int_{0}^{\pi} \sin nx\,dx = 0 - \frac{1}{\pi}\frac{1}{n}\left[\cos nx\right]_0^{\pi} =$$

$$= -\frac{1}{\pi n}\left[\cos n\pi - \cos 0\right] = -\frac{1}{\pi n}\left[(-1)^n - 1\right] = \begin{cases} n \text{ sudé } n = 2k & 0 \\ n \text{ liché } n = 2k-1 & \frac{2}{\pi n} = \frac{2}{\pi(2k-1)} \end{cases}$$

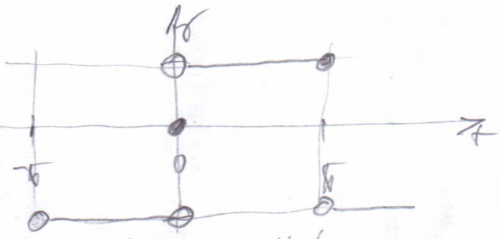
$$f(x) = a_0 + a_1\cos x + b_1\sin x + a_2\cos 2x + b_2\sin 2x + \dots + =$$

$$= \frac{1}{2} + \underbrace{0 + \frac{2}{\pi}\sin x}_{k=1} + \underbrace{0 + \frac{2}{3\pi}\sin 3x}_{k=2} + \underbrace{0 + \frac{2}{5\pi}\sin 5x}_{k=3} + \dots$$

$$f(x) = \frac{1}{2} + \sum_{k=1}^{\infty}\frac{2}{(2k-1)\pi}\sin[(2k-1)x] \Rightarrow S_n(x) = \frac{1}{2} + \frac{2}{\pi}\sin\pi + \frac{2}{3\pi}\sin 3\pi + \dots + \frac{2}{n\pi}\sin nx$$

viz obr.

(Př 5) Nalezněte FŘ fce $f(t) = \begin{cases} 1; & t \in (0;\pi\rangle \\ 0; & t = 0 \\ -1; & t \in (-\pi;0) \end{cases}$



$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt = \frac{1}{2\pi}\left\{\int_{-\pi}^{0}(-1)\,dt + \int_{0}^{0} 0\,dt + \int_{0}^{\pi} 1\,dt\right\} =$$

$$= \frac{1}{2\pi}\left\{\left[-t\right]_{-\pi}^{0} + 0 + \left[t\right]_0^{\pi}\right\} = \frac{1}{2\pi}\left\{\left[(-0) - (+\pi)\right] + 0 + \left[(\pi) - 0\right]\right\} =$$

$$= \frac{1}{2\pi}\left\{0 - \pi + 0 + \pi + 0\right\} = \underline{0}$$

po rozšíření spojitá na $\langle -\pi;\pi\rangle$ fce lichá $\Rightarrow a_n = 0$ $n = 0, 1 \dots$

$T = 2\pi \Rightarrow \omega = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt\,dt = \frac{1}{\pi}\left\{\int_{-\pi}^{0}(-1)\cos nt\,dt + 0 + \int_{0}^{\pi} 1\cos nt\,dt\right\} = \frac{1}{\pi}\left\{-\left[\frac{1}{n}\sin nt\right]_{-\pi}^{0} + \left[\frac{1}{n}\sin nt\right]_0^{\pi}\right\} =$$

$$= \frac{1}{\pi}\left\{-\left[\cancel{\frac{1}{n}\sin 0} - \cancel{\frac{1}{n}\sin 0}\right] + \left[\cancel{\frac{1}{n}\sin n\pi} - \cancel{\frac{1}{n}\sin 0}\right]\right\} = 0$$

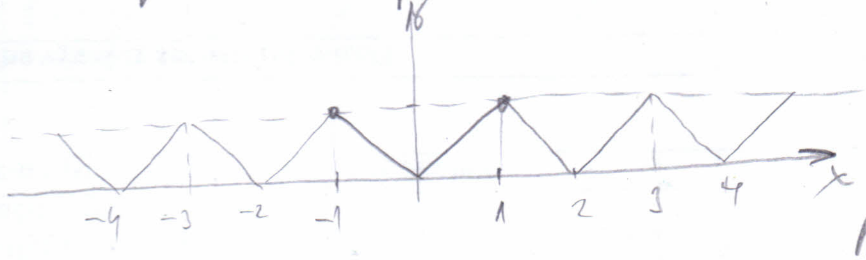
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin nt - \frac{1}{\pi}\left\{\int_{-\pi}^{0}(-1)\sin nt\,dt + 0 + \int_{0}^{\pi} 1\sin nt\,dt\right\} =$$

$$= \frac{1}{\pi}\left\{\left[\frac{1}{n}\cos nt\right]_{-\pi}^{0} - \left[\frac{1}{n}\cos nt\right]_{0}^{\pi}\right\} = \frac{1}{\pi n}\left\{\cos 0 - \cos n\pi - \cos n\pi + \cos 0\right\} =$$

$$= \frac{1}{\pi n}\left[1-(-1)^n-(-1)^n+1\right] = \frac{1}{\pi n}\left[2-2(-1)^n\right] = \frac{2}{\pi n}\left[1-(-1)^n\right] = \begin{cases} 0 & n \text{ sudé } n=2k \\ \frac{4}{\pi n} & n \text{ liché } n=2k-1 \end{cases}$$

$$f(t) = 0 + \sum_{k=1}^{\infty}\frac{4}{\pi(2k-1)}\sin\left[(2k-1)t\right] = \underset{(k=1)}{\frac{4}{\pi}\sin t} + \underset{(k=2)}{\frac{4}{3\pi}\sin 3t} + \underset{(k=3)}{\frac{4}{5\pi}\sin 5t} + \dots$$

Př. 6) Nalezněte FŘ pro Δ vlnu $f(x) = |x|$ pro $-1 \le x \le 1$ a $f(x+2) = f(x)$ $\forall x$



fce je spojitá a sudá $\Rightarrow b_n = 0$

Perioda je $T=2 \Rightarrow L=1$

$\omega = \frac{2\pi}{T} = \frac{2\pi}{2} = \pi$

$$a_0 = \frac{1}{2}\int_{-1}^{1}|x|\,dx = \frac{1}{2}\int_{-1}^{0}(-x)\,dx + \frac{1}{2}\int_{0}^{1}x\,dx = -\frac{1}{2}\left[\frac{x^2}{2}\right]_{-1}^{0} + \frac{1}{2}\left[\frac{x^2}{2}\right]_{0}^{1} = -\frac{1}{4}\left[(0-1)\right] + \frac{1}{4}\left[1-0\right] =$$

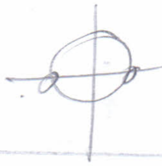
$$= \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

$$a_n = \frac{1}{1}\int_{-1}^{1}|x|\cos(n\pi x)\,dx = -\int_{-1}^{0}x\cos n\pi x\,dx + \int_{0}^{1}x\cos n\pi x\,dx = -\left[\frac{x}{n\pi}\sin n\pi x + \frac{1}{\pi^2 n^2}\cos n\pi x\right]_{-1}^{0} +$$

$$\int x\cos n\pi x\,dx = \frac{x}{n\pi}\sin n\pi x - \frac{1}{\pi n}\int \sin n\pi x\,dx = \frac{x}{n\pi}\sin n\pi x + \frac{1}{\pi^2 n^2}\cos n\pi x$$

$u = x \quad u' = 1$

$v' = \cos n\pi x \quad v = \frac{1}{n\pi}\sin n\pi x$



$$+\left[\frac{x}{n\pi}\sin n\pi x + \frac{1}{\pi^2 n^2}\cos n\pi x\right]_{0}^{1} = -\left[\left(\cancel{\frac{0}{\pi n}\sin 0} + \frac{1}{\pi^2 n^2}\cos 0\right) - \left(\cancel{\frac{-1}{n\pi}\sin(-n\pi)} + \frac{1}{\pi^2 n^2}\cos(-n\pi)\right)\right] +$$

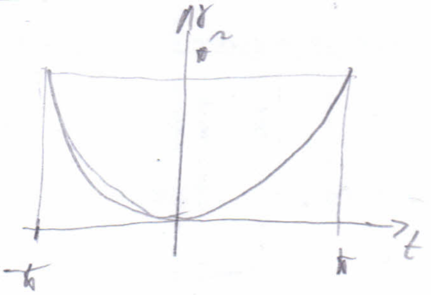
$$+\left[\left(\cancel{\frac{0}{n\pi}\sin 0} + \frac{1}{\pi^2 n^2}\cos 0\right) - \left(\cancel{\frac{-1}{n\pi}\sin(-n\pi)} + \frac{1}{\pi^2 n^2}\cos(-n\pi)\right)\right] = -\left(\frac{1}{\pi n^2} - \frac{1}{\pi^2 n^2}(-1)^n + \frac{1}{\pi^2 n^2} - \frac{1}{\pi^2 n^2}(-1)^n\right)$$

$$= -\left(\frac{2}{\pi^2 n^2} - \frac{2}{\pi^2 n^2}(-1)^n\right) = -\frac{2}{\pi^2 n^2}\left(1-(-1)^n\right) = \begin{cases} 0 & n \text{ sudé } n=2k \\ -\frac{4}{\pi^2 n^2} & n \text{ liché } n=2k-1 \end{cases} \quad \left(-\frac{4}{\pi^2(2k-1)^2}\right.$$

$$f(x) = \frac{1}{2} - \sum_{k=1}^{\infty} \frac{4}{\pi^2(2k-1)^2}\cos[(2k-1)\pi x] = \underbrace{\frac{1}{2} - \frac{4}{\pi^2}\cos(\pi x)}_{k=1} \underbrace{- \frac{4}{9\pi^2}\cos(3\pi x)}_{k=2} \underbrace{- \frac{4}{25\pi^2}\cos(5\pi x)}_{k=3} - \dots$$

celkom: $$|x| = \frac{1}{2} - \sum_{k=1}^{\infty} \frac{4}{(2k-1)^2\pi^2}\cos[(2k-1)\pi x]; \quad x \in \langle -1;1\rangle$$

Př 7 $f(t) = t^2$ $\langle -\pi, \pi\rangle$



fce spojitá sudá $\Rightarrow b_n = 0$

$T = 2\pi$ $\omega = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} t^2\,dt = \frac{2}{2\pi}\int_0^{\pi} t^2\,dt = \frac{1}{\pi}\left[\frac{t^3}{3}\right]_0^{\pi} = \frac{1}{\pi}\frac{\pi^3}{3} = \frac{\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} t^2\cos nt\,dt = \frac{2}{\pi}\int_0^{\pi} t^2 \cos nt\,dt = \frac{2}{\pi}\left[\frac{t^2}{n}\sin nt + \frac{2t}{n^2}\cos nt - \frac{2}{n^3}\sin nt\right]_0^{\pi} \ominus$$

$$\int t^2 \cos nt\,dt = \frac{t^2}{n}\sin nt - \frac{2}{n}\int t \sin nt = \frac{t^2}{n}\sin nt - \frac{2}{n}\left(-\frac{t}{n}\cos nt + \frac{1}{n}\int \cos nt\,dt\right) =$$

$u = t^2$ $u' = 2t$
$v' = \cos nt$ $v = \frac{1}{n}\sin nt$

$u = t$ $u' = 1$
$v' = \sin nt$ $v = -\frac{1}{n}\cos nt$

$$= \frac{t^2}{n}\sin nt + \frac{2t}{n^2}\cos nt - \frac{2}{n^3}\sin nt$$

$$\ominus\ \frac{2}{\pi}\left[\left(\frac{\pi^2}{n}\cancel{\sin n\pi} + \frac{2\pi}{n^2}\cos n\pi - \frac{2}{n^3}\cancel{\sin n\pi}\right) - \left(0 + 0 - \frac{2}{n^3}\cancel{\sin 0}\right)\right] = \frac{2}{\pi}\cdot\frac{2\pi}{n^2}\cos n\pi =$$

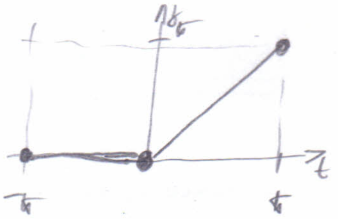
$$= \frac{4}{n^2}(-1)^n$$

$t \in \langle -\pi, \pi\rangle$

$$f(t) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4}{n^2}(-1)^n \cos nt = \frac{\pi^2}{3} \underbrace{- 4\cos t}_{n=1} + \underbrace{\cos 2t}_{n=2} \underbrace{- \frac{4}{9}\cos 3t}_{n=3} + \underbrace{\frac{1}{4}\cos 4t}_{n=4} \underbrace{- \frac{4}{25}\cos 5t}_{n=5} + \dots$$

Př 3

$$f(t) = \begin{cases} 0 & t \in \langle -\pi ; 0) \\ t & t \in \langle 0 ; \pi \rangle \end{cases}$$



fce je spojitá
není L ani S
$T = 2\pi \quad \omega = \frac{2\pi}{2\pi} = 1$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,dt = \frac{1}{\pi}\left(\int_{-\pi}^{0} 0\,dt + \int_{0}^{\pi} t\,dt\right) = \frac{1}{\pi}\left[\frac{t^2}{2}\right]_0^{\pi} = \frac{1}{\pi}\frac{\pi^2}{2} = \frac{\pi}{2}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt\,dt = \frac{1}{\pi}\left\{\int_{-\pi}^{0} 0\cos nt\,dt + \int_{0}^{\pi} t\cos nt\,dt\right\} = \frac{1}{\pi}\int_{0}^{\pi} t\cos nt\,dt =$$

$u = t \quad u' = 1$

$v' = \cos nt \quad v = \frac{1}{n}\sin nt$

$$= \frac{1}{\pi}\left\{\left[\frac{t}{n}\sin nt\right]_0^{\pi} - \frac{1}{n}\int_0^{\pi}\sin nt\,dt\right\} = \frac{1}{\pi}\left\{\left[\frac{t}{n}\sin nt\right]_0^{\pi} + \frac{1}{n^2}\left[\cos nt\right]_0^{\pi}\right\} =$$

$$= \frac{1}{\pi}\left\{\left[\Big(\frac{\pi}{n}\underbrace{\sin n\pi}_{0}\Big) - \underbrace{\frac{0}{n}\sin 0}_{0}\right] + \frac{1}{n^2}\left[(\cos n\pi) - \cos 0\right]\right\} = \frac{1}{\pi n^2}\left[(-1)^n - 1\right] = \begin{cases} n=2k & 0 \\ n=2k-1 & -\frac{2}{\pi n^2} \end{cases}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin nt\,dt = \frac{1}{\pi}\left\{\int_{-\pi}^{0} 0\sin nt\,dt + \int_{0}^{\pi} t\sin nt\,dt\right\} = \frac{1}{\pi}\int_0^{\pi} t\sin nt\,dt =$$

$u = t \quad u' = 1$

$v' = \sin nt \quad v = -\frac{1}{n}\cos nt$

$$= \frac{1}{\pi}\left\{\left[-\frac{t}{n}\cos nt\right]_0^{\pi} + \frac{1}{n}\int_0^{\pi}\cos nt\,dt\right\} = \frac{1}{\pi}\left\{\left[\Big(-\frac{\pi}{n}\cos n\pi\Big) - 0\right] + \frac{1}{n^2}\left[\sin nt\right]_0^{\pi}\right\} =$$

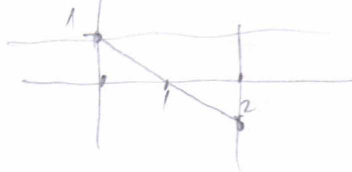
$$= \frac{1}{\pi}\left\{-\frac{\pi}{n}(-1)^n + \cancel{\frac{1}{n^2}\left[(\underbrace{\sin n\pi}_{0}) - \underbrace{\sin 0}_{0}\right]}\right\} = -\frac{\pi}{\pi n}\left[(-1)^n\right] = \begin{cases} n=2k & -\frac{1}{n} \\ n=2k-1 & \frac{1}{n} \end{cases}$$

$t \in \langle -\pi ; \pi \rangle$

$$f(t) = \frac{\pi}{4} + \sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{\pi n^2}\cos nt - \frac{(-1)^n}{n}\sin nt\right] = \frac{\pi}{4} + \underbrace{\left(-\frac{2}{\pi}\cos t + \sin t\right)}_{n=1} + \underbrace{\left(0 - \frac{1}{2}\sin 2t\right)}_{n=2} +$$

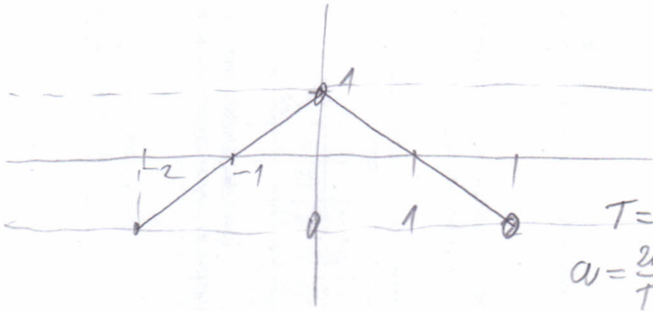
$$+ \underbrace{\left(-\frac{2}{9\pi}\cos 3t + \frac{1}{3}\sin 3t\right)}_{n=3} + \underbrace{\left(0 - \frac{1}{4}\sin 4t\right)}_{n=4} + \underbrace{\left(-\frac{2}{25\pi}\cos 5t + \frac{1}{5}\sin 5t\right)}_{n=5} + \underbrace{\left(0 - \frac{1}{6}\sin 6t\right)}_{n=6} + \ldots$$

Př. 9) Profi $f(t) = 1-t$, $t \in \langle 0;2 \rangle$ urči



a) její rozvoj v kosinovou řadu

$\rightarrow$ fci $f(t) = 1-t$ rozšíříme na $f^*(t)$ tak, aby jsme dostali sudou fci na $\langle -2;2 \rangle$



$T = 4 \quad L = 2$

$\omega = \frac{2\pi}{T} = \frac{2\pi}{4} = \frac{\pi}{2}$

$$a_0 = \frac{1}{4}\int_{-2}^{2}(1-t)\,dt = \frac{2}{4}\int_{0}^{2}(1-t)\,dt = \frac{1}{2}\left[t - \frac{t^2}{2}\right]_0^2 = \frac{1}{2}\left[\left(2-\frac{4}{2}\right)-(0-0)\right] = 0$$

$$a_n = \frac{2}{4}\int_{-2}^{2}(1-t)\cos\frac{n\pi t}{2}\,dt = \frac{4}{4}\int_{0}^{2}(1-t)\cos\frac{n\pi t}{2}\,dt = \left[\frac{2(1-t)}{n\pi}\sin\frac{n\pi t}{2} - \frac{4}{n^2\pi^2}\cos\frac{n\pi t}{2}\right]_0^2 \ominus$$

$$\int(1-t)\cos\frac{n\pi t}{2}\,dt = \frac{2(1-t)}{n\pi}\sin\frac{n\pi t}{2} + \frac{2}{n\pi}\int\sin\frac{n\pi t}{2}\,dt = \frac{2(1-t)}{n\pi}\sin\frac{n\pi t}{2} - \frac{4}{n^2\pi^2}\cos\frac{n\pi t}{2}$$

$u = 1-t \quad u' = -1$

$v' = \cos\frac{n\pi t}{2} \quad v = \frac{2}{n\pi}\sin\frac{n\pi t}{2}$

$$\ominus \left[\left(\cancel{\frac{2(1-2)}{2n}\sin n\pi} - \frac{4}{n^2\pi^2}\cos n\pi\right) - \left(\cancel{\frac{2}{n\pi}\sin 0} - \frac{4}{n^2\pi^2}\cos 0\right)\right] = -\frac{4}{n^2\pi^2}(-1)^n + \frac{4}{\pi^2 n^2} =$$

$$= \frac{4}{\pi^2 n^2}\left(1-(-1)^n\right) = \begin{cases} n = \text{sudé } n = 2k & 0 \\ n \text{ liché } n = 2k-1 & \frac{8}{\pi^2 n^2} \end{cases}$$

FŘ $t \in \langle 0;2 \rangle$

$$1 - t = \sum_{k=1}^{\infty}\left[\frac{8}{\pi^2(2k-1)^2}\cos\frac{(2k-1)\pi t}{2}\right] = \underbrace{\left(\frac{8}{\pi^2}\cos\frac{\pi t}{2}\right)}_{k=1} + \underbrace{\left(\frac{8}{9\pi^2}\cos\frac{3\pi t}{2}\right)}_{k=2} + \underbrace{\left(\frac{8}{25\pi^2}\cos\frac{5\pi t}{2}\right)}_{k=3} + \dots$$

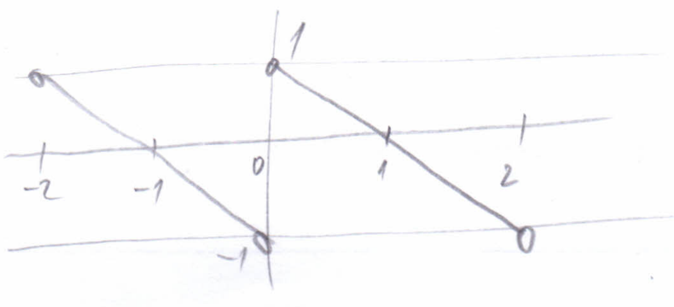
b) její rozvoj v sinovou řadu

$\rightarrow$ $f(t) = 1-t$ rozšíříme na $f^{**}(t)$ tak, aby jsme dostali lichou fci na $\langle -2;2 \rangle$

$a_n = 0$

$T = 4 \quad L = 2$

$\omega = \frac{2\pi}{T} = \frac{2\pi}{4} = \frac{\pi}{2}$



$$b_n = \frac{4}{4}\int_0^2 (1-t)\sin\frac{n\pi t}{2}\,dt = \int_0^2 (1-t)\sin\frac{n\pi t}{2}\,dt = \left[\frac{2(t-1)}{n\pi}\cos\frac{n\pi t}{2} - \frac{4}{n^2\pi^2}\sin\frac{n\pi t}{2}\right]_0^2 \ominus$$

$$\int (1-t)\sin\frac{n\pi t}{2}\,dt = \frac{2(t-1)}{n\pi}\cos\frac{n\pi t}{2} - \frac{2}{n\pi}\int\cos\frac{n\pi t}{2}\,dt = \frac{2(t-1)}{n\pi}\cos\frac{n\pi t}{2} - \frac{4}{n^2\pi^2}\sin\frac{n\pi t}{2}$$

$u = 1-t \qquad u' = -1$

$v' = \sin\frac{n\pi t}{2} \qquad v = -\frac{2}{n\pi}\cos\frac{n\pi t}{2}$

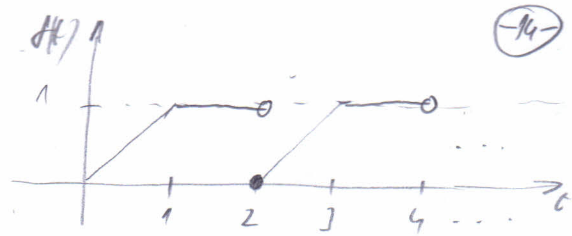
$$\ominus \left[\left(\frac{2}{n\pi}\cdot\underbrace{\cos n\pi}_{(-1)^n} - \cancel{\frac{4}{n^2\pi^2}\underbrace{\sin n\pi}_{0}}\right) - \left(\frac{-2}{n\pi}\underbrace{\cos 0}_{1} - \cancel{\frac{4}{n^2\pi^2}\underbrace{\sin 0}_{0}}\right)\right] = \left[\frac{2}{n\pi}\cdot(-1)^n + \frac{2}{n\pi}\right] =$$

$$= \frac{2}{n\pi}\left[(-1)^n + 1\right] = \begin{cases} n = 2k: & \frac{4}{n\pi} \\ n = 2k-1: & 0 \end{cases}$$

FŘ pro $t \in \langle 0; 2 \rangle$:

$$1-t = \sum_{k=1}^{\infty}\left(\frac{4}{2k\pi}\sin\frac{2k\pi t}{2}\right) = \sum_{k=1}^{\infty}\left(\frac{2}{k\pi}\sin k\pi t\right) = \underbrace{\frac{2}{\pi}\sin\pi t}_{k=1} + \underbrace{\frac{1}{\pi}\sin 2\pi t}_{k=2} + \underbrace{\frac{2}{3\pi}\sin 3\pi t}_{k=3} + \underbrace{\frac{1}{2\pi}\sin 4\pi t}_{k=4} + \ldots$$

(Př1) Urči FŘ fce $f(t) = \begin{cases} t & t\in \langle 0;1) \\ 1 & t\in\langle 1;2)\end{cases}$



$T=2$ ŠK

$$f(t)\sim \frac{a_0}{2}+\sum_{k=1}^{\infty} (a_k\cos k\omega t + b_k \sin k\omega t) \Longleftarrow T=\frac{2\pi}{\omega} \Rightarrow \omega = \frac{2\pi}{T}=\frac{2\pi}{2}=\pi$$

kde: $a_0 = \frac{2}{T}\int_0^T f(t)\cdot\cos 0\,dt = \frac{2}{2}\int_0^2 f(t)\,dt = 1\int_0^1 t\,dt + \int_1^2 1\,dt = \left[\frac{t^2}{2}\right]_0^1 + [t]_1^2 = \left(\frac12-0\right)+(2-1)=\frac32$

$a_k = \frac2T\int_0^T f(t)\cdot\cos k\pi t\,dt = 1\cdot\int_0^1 t\cdot\cos k\pi t\,dt + \int_1^2 1\cdot\cos k\pi t\,dt \ominus$

$\sin k\pi=0$
$\cos k\pi = (-1)^k$
$\sin 2k\pi=0$
$\cos 2k\pi = 1$

$\int t\cos k\pi t\,dt = \frac{t}{k\pi}\cdot\sin k\pi t - \frac{1}{k\pi}\int \sin k\pi t\,dt = \frac{t}{k\pi}\sin k\pi t + \frac{1}{k^2\pi^2}\cdot\cos k\pi t$

$u=t \quad u'=1$
$v'=\cos k\pi t \quad v = \frac{1}{k\pi}\sin k\pi t$

$\ominus \left[\frac{t}{k\pi}\sin k\pi t\right]_0^1 + \left[\frac{1}{k^2\pi^2}\cos k\pi t\right]_0^1 + \left[\frac1{k\pi}\sin k\pi t\right]_1^2 = \left(\frac{1}{k\pi}\overset{0}{\sin k\pi} - 0\right) + \left(\frac{1}{k^2\pi^2}\cos k\pi - \frac1{k^2\pi^2}\cos 0\right) +$
$+\left(\frac1{k\pi}\sin 2k\pi - \frac1{k\pi}\sin k\pi\right) = 0 + \frac{1}{k^2\pi^2}(-1)^k - \frac{1}{k^2\pi^2} + 0 = \frac{1}{k^2\pi^2}\left[(-1)^k-1\right] = \begin{cases} 0 & k \text{ sudé } k=2l \\ -\frac{2}{k^2\pi^2} & k \text{ liché } k=2l+1 \end{cases}$

$b_k = \frac2T\int_0^T f(t)\cdot\sin k\pi t\,dt = 1\int_0^1 t\sin k\pi t\,dt + \int_1^2 1\cdot\sin k\pi t\,dt \ominus$

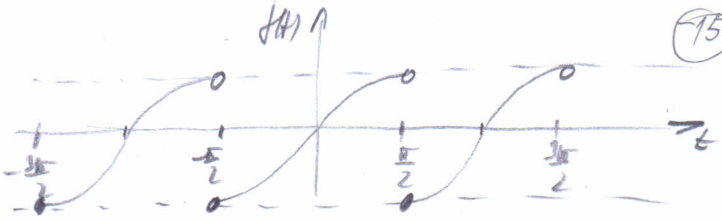
$\int t\sin k\pi t\,dt = -\frac{t}{k\pi}\cos k\pi t + \frac1{k\pi}\int\cos k\pi t\,dt = -\frac{t}{k\pi}\cos k\pi t + \frac{1}{k^2\pi^2}\sin k\pi t$

$t=u \quad t'=1$
$u'=\sin k\pi t \quad u = -\frac{1}{k\pi}\cos k\pi t$

$\ominus \left[-\frac{t}{k\pi}\cos k\pi t\right]_0^1 + \left[\frac{1}{k^2\pi^2}\sin k\pi t\right]_0^1 + \left[-\frac1{k\pi}\cos k\pi t\right]_1^2 = \left(-\frac1{k\pi}\cos k\pi + 0\right) + \left(\frac{1}{k^2\pi^2}\overset{0}{\sin k\pi} - \cancel{\frac1{k^2\pi^2}\sin 0}\right) +$
$-\frac{1}{k\pi}\left(\underset{1}{\cos 2k\pi} - \cos k\pi\right) = -\frac{1}{k\pi}(-1)^k - \frac1{k\pi} + \frac1{k\pi}(-1)^k = -\frac{1}{k\pi}\left[\cancel{(-1)^k}+1-\cancel{(-1)^k}\right] = -\frac{1}{k\pi}$

$$f(t)\sim \frac34 + \sum_{k=1}^{\infty} \left[\frac{(-1)^k-1}{k^2\pi^2}\cos k\pi t - \frac1{k\pi}\sin k\pi t\right]$$ (v bodech nespojitosti $t=2k$ tam je součet $\frac12$)

Př 11) $f(t) = \sin t \qquad t \in \langle -\frac{\pi}{2}; \frac{\pi}{2})$



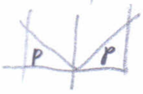
$$f(t) \sim \cancel{\frac{a_0}{2}} + \sum_{k=1}^{\infty} \left(\cancel{a_k \cos 2kt} + b_k \sin 2kt\right)$$

lichá fce $\quad a_0 = 0 \quad a_k = 0$

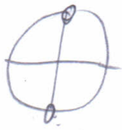
$T = \pi \qquad T = \frac{2\pi}{\omega} \Rightarrow \omega = \frac{2\pi}{T} = \frac{2\pi}{\pi} = 2$

$$\int_{-a}^{a} \text{lichá fce} = -P + P = 0$$

$$\int_{-a}^{a} \text{sudá fce} = P + P = 2P$$



$$a_0 = \frac{2}{T}\int_0^T f(t)\cdot 1\,dt = \frac{2}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin t\,dt = \frac{2}{\pi}\left[-\cos t\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{2}{\pi}\left[-\cos\frac{\pi}{2} + \cos\left(-\frac{\pi}{2}\right)\right] = 0$$



$\sin\left[(1-2k)\frac{\pi}{2}\right] = (-1)^k$

$\sin\left[(1+2k)\frac{\pi}{2}\right] = (-1)^k$

$a_k = 0$ lichá fce

$$b_k = \frac{2}{T}\int_0^T f(t)\cdot \sin 2kt\,dt = \frac{2}{\pi}\cdot 2\int_0^{\frac{\pi}{2}} \sin t \cdot \sin 2kt\,dt \ominus$$

$$\int \sin t \cdot \sin 2kt\,dt = \left|\begin{array}{l} \text{GON. VZORCE} \\ \sin\alpha\cdot\sin\beta = \frac{1}{2}\left[\cos(\alpha-\beta) - \cos(\alpha+\beta)\right] \\ \cos\alpha\cdot\cos\beta = \frac{1}{2}\left[\cos(\alpha+\beta) + \cos(\alpha-\beta)\right] \\ \sin\alpha\cdot\cos\beta = \frac{1}{2}\left[\sin(\alpha+\beta) + \sin(\alpha-\beta)\right] \end{array}\right| = \int \frac{1}{2}\left[\cos(t-2kt) - \cos(t+2kt)\right]dt =$$

$$= \frac{1}{2}\int \cos\left[t(1-2k)\right]dt - \frac{1}{2}\int\cos\left[t(1+2k)\right]dt =$$

$$= \frac{1}{2}\,\frac{1}{1-2k}\sin\left[t(1-2k)\right] - \frac{1}{2}\,\frac{1}{1+2k}\sin\left[t(1+2k)\right]$$

$$\ominus\ \frac{4}{\pi}\left\{\left[\frac{1}{2}\cdot\frac{1}{1-2k}\sin\left[t(1-2k)\right]\right]_0^{\frac{\pi}{2}} - \left[\frac{1}{2}\,\frac{1}{1+2k}\sin\left[t(1+2k)\right]\right]_0^{\frac{\pi}{2}}\right\} = \frac{4}{\pi}\left\{\left[\left(\frac{1}{2}\cdot\frac{1}{1-2k}\sin\left[(1-2k)\frac{\pi}{2}\right] - 0\right) + \right.\right.$$

$$\left.\left. - \left[\left(\frac{1}{2}\,\frac{1}{1+2k}\sin\left[(1+2k)\frac{\pi}{2}\right]\right) - 0\right]\right]\right\} = \frac{4}{\pi}\left[\frac{1}{2}\,\frac{1}{1-2k}(-1)^k - \frac{1}{2}\,\frac{1}{1+2k}\cdot(-1)^k\right] = \frac{\overset{2}{\cancel{4}}}{\pi}\,\cancel{\frac{1}{2}}\cdot(-1)^k\left[\frac{1}{1-2k} - \frac{1}{1+2k}\right] =$$

$$= \frac{2}{\pi}(-1)^k\left[\frac{1+2k-1+2k}{(1-2k)(1+2k)}\right] = \frac{2}{\pi}(-1)^k\cdot\frac{4k}{1-4k^2}$$

$$f(t) \sim \frac{8}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^k k}{1-4k^2}\sin 2kt \qquad t \in \left(-\frac{\pi}{2}+k\pi; \frac{\pi}{2}+k\pi\right)$$

$$\sim 0 \qquad t = \frac{\pi}{2}+k\pi$$

→ když je $f(t)$ lichá fce → $a_k = 0$; počítáme pouze b_k → sinová řada

$f(t)$ sudá fce → $b_k = 0$; počítáme pouze a_0 a a_k → kosinová řada

a platí: $a_k = \frac{2}{T}\cdot 2\int_0^{\frac{T}{2}} f(t)\cos k\omega t\,dt$

$b_k = \frac{2}{T}\cdot 2\int_0^{\frac{T}{2}} f(t)\sin k\omega t\,dt$

→ když periodické opakování impulsu f_t na $\langle 0;T)$ není lichá fce, a přesto bychom chtěli vytvořit sinovou řadu

→ musíme udělat liché prodloužení impulsu na interval $\langle -T;T)$

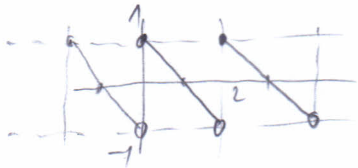
→ jeho opakováním vznikne fce s periodou $2T$, jiná než ta původní, ale na intervalu $\langle 0;T)$ se bude shodovat s původním impulsem f_t

→ při počítání koeficientů $b_k = \frac{2}{(2T)} \cdot \int_{-T}^{T} \underbrace{\text{prodloužený impuls}}_{\text{lichá}\cdot\text{lichá}=\text{sudá}} \cdot \sin k\omega t\, dt \ominus$ $\quad \omega = \frac{2\pi}{(2T)}$

$$\ominus \frac{2}{(2T)} \cdot 2 \int_0^T f_T(t) \cdot \sin k\omega t\, dt$$

⇒ nemusíme otrocovat předpis prodloužení na $\langle -T;0)$

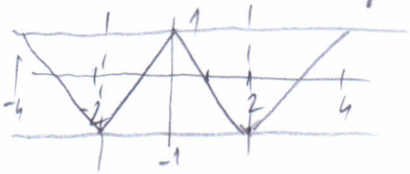
(Př 12) Najdi kosinovou řadu pro $f_T(t) = 1-t$; $t \in \langle 0;2)$.



perioda $T=2$ je lichá fce ⇒ vyšla by sinová řada

$\omega = \frac{2\pi}{T} = \frac{2\pi}{2} = \pi \qquad f \approx \sum_{k=1}^{\infty} b_k \sin k\pi t$

→ musíme nejdříve udělat sudé prodloužení impulsu na interval $(-2;2)$



s periodou $T' = 4$ je sudá fce → vyjde kosinová řada

$\omega' = \frac{2\pi}{T'} = \frac{2\pi}{4} = \frac{\pi}{2}$

→ řada pro $f \approx \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos k\frac{\pi}{2} t$

$$a_0 = \frac{2}{4} \cdot 2 \int_0^2 (1-t)\, dt = 1 \left[t - \frac{t^2}{2}\right]_0^2 = 2-2 = 0$$

$$a_k = \frac{2}{4} \cdot 2 \int_0^2 (1-t) \cos k\frac{\pi}{2} t\, dt = \left[\frac{2(1-t)}{k\pi} \sin k\frac{\pi}{2} t\right]_0^2 - \frac{4}{k^2\pi^2}\left[\cos k\frac{\pi}{2} t\right]_0^2 \ominus$$

$$\int (1-t) \cos k\frac{\pi}{2} t\, dt = \frac{2(1-t)}{k\pi} \sin k\frac{\pi}{2} t + \frac{2}{k\pi} \cdot \int \sin k\frac{\pi}{2} t\, dt = \frac{2(1-t)}{k\pi} \sin k\frac{\pi}{2} t - \frac{4}{k^2\pi^2} \cos k\frac{\pi}{2} t$$

$u = 1-t \quad u' = -1$

$v' = \cos k\frac{\pi}{2} t \quad v = \frac{2}{k\pi} \sin k\frac{\pi}{2} t$

$\oplus$ $\sin k\pi = 0$, $\cos k\pi = (-1)^k$

$$\ominus \left[\left(-\frac{2}{k\pi} \cancel{\sin \pi k}\right) - \left(\frac{2}{k\pi} \cancel{\sin 0}\right)\right] - \frac{4}{k^2\pi^2}\left[(\cos \pi k - \cos 0)\right] = -\frac{4}{k^2\pi^2}\left[(-1)^k - 1\right] = \begin{cases} 0 & k \text{ sudé } k=2l \\ \frac{8}{k^2\pi^2} & k \text{ liché } k=2l+1 \end{cases}$$

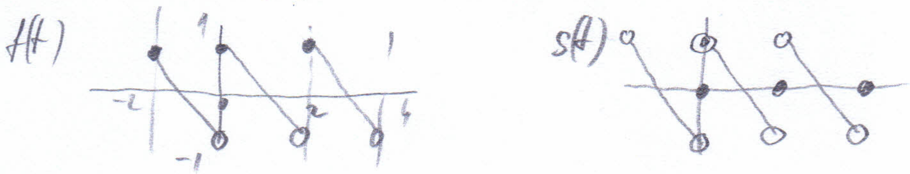
kosinová řada $\sum_{l=1}^{\infty} \frac{8}{(2l+1)^2\pi^2} \cos\left[(2l+1) \cdot \frac{\pi}{2} t\right]$

součet $s(t)$ = periodické opakování sudého prodlouženého impulsu s periodou $T' = 4$

→ Sinová řada pro $f_T = 1-t$; $t \in \langle 0;2)$

vyjde stejně jako normální FŘ, neboť periodické opakování je lichá fce

$$f_T \approx \sum_{k=1}^{\infty} \frac{2}{k\pi} \sin k\pi t$$



ověřte

→ FŘ pro $f(t) = \sin t$; $t \in \mathbb{R}$

→ je s periodou $T = 2\pi \to \omega = 1$

→ fce je lichá → řada bude sinová

→ $f \sim \sum_{k=1}^{\infty} b_k \sin k \cdot 1 \cdot t$ vyjde $b_1 = 1$

$b_k = 0$ pro $k \geq 2$

$$f(t) = \sin t \approx 1 \sin t + 0 \cdot \sin 2t + 0 \cdot \ldots$$

→ FŘ pro $f(t) = \sin^2 t$

$$\sin^2 t = \frac{1-\cos 2t}{2} = \frac{1}{2} - \frac{1}{2}\cos 2t$$

$\updownarrow$

toto je kosinová FR pro $\sin^2 t$

tj: $a_0 = 1$; $a_2 = -\frac{1}{2}$; $a_k = 0$ pro $k \geq 3$